

QUADRATIC BASE CHANGE AND THE ANALYTIC CONTINUATION OF THE ASAI L-FUNCTION: A NEW TRACE FORMULA APPROACH

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ABSTRACT. Using Langlands's *Beyond Endoscopy* idea and analytic number theory techniques, we study the Asai L-function associated to a real quadratic field \mathbf{K}/\mathbb{Q} . If the Asai L-function associated to an automorphic form over \mathbf{K} has a pole, then the form is a base change from \mathbb{Q} . We prove this and further prove the analytic continuation of the L-function. This is one of the first examples of using a trace formula to get such information. A hope of Langlands is that general L-functions can be studied via this method.

1. INTRODUCTION

Understanding the analytic continuation of L-functions is a central object of interest in number theory. If the L-function is associated to an automorphic form, then the analytic properties of the function are key in understanding the form itself. This is clearly demonstrated with the Asai L-function in this paper. We will now define it.

Let $\mathbf{K} = \mathbb{Q}(\sqrt{D})$ be a real quadratic extension of \mathbb{Q} , with D a prime. We assume for computational clarity in this paper that \mathbf{K} has class number one. Let Π be an automorphic form with level one and trivial nebentypus over the field \mathbf{K} . The details we need of a automorphic form we save to next section, and for more elaborate details we refer to [BMP1], [BMP2], and [V]. If it is more comfortable, one can think of a Hilbert modular form instead of an automorphic form over \mathbf{K} . The form has associated to it Fourier coefficients $\{c_\mu(\Pi)\}$ parametrized by integral ideals μ .

The standard L-function associated with the form is

$$L(s, \Pi) = \sum_{\mu} \frac{c_\mu(\Pi)}{\mathbb{N}(\mu)^s}.$$

It is well known $L(s, \Pi)$ has analytic continuation to the complex plane and satisfies a functional equation if Π is a cuspidal automorphic form.

The Asai L-function for Π is a sort of subseries of $L(s, \Pi)$. We define the series as

$$L(s, \Pi, \text{Asai}) = \zeta(2s) \sum_{n=1}^{\infty} \frac{c_n(\Pi)}{n^s}.$$

The L-function is named after Asai, who in [A] showed the function has analytic continuation, up to a pole at $s = 1$, an Euler product, and a functional equation. Further, if it has a pole then the associated form is a base change or a lift from a automorphic form over \mathbb{Q} . Base change is usually denoted $\Pi = BC_{\mathbf{K}/\mathbb{Q}}(\pi)$, where π is an automorphic form over \mathbb{Q} . In terms of L-functions, which is easier for the author to think about, the Asai L-function having a pole is equivalent to the decomposition of the standard L-function as product of 2 degree 2

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L-functions over \mathbb{Q} . Namely,

$$L(s, \Pi) = L(s, \pi)L(s, \pi \otimes \chi_D),$$

where χ_D is the character associated to the field $\mathbf{K} = \mathbb{Q}(\sqrt{D})$. A key point is that even if Π is a base change its standard L-function is entire, just not primitive. The Asai L-function, on the other hand, "detects" or sieves those forms for us!

To study these forms and their associated Asai L-function we use the Kuznetsov trace formula. We use a technique originally formulated in [L], and modified in [S], to average over the spectrum of cuspidal automorphic forms over the quadratic field \mathbf{K} , along with an averaging over the Fourier coefficients. In other words, we study (ignoring test functions and convergence issues of the Π -sum)

$$(1.1) \quad \frac{1}{X} \sum_{\mu} g\left(\frac{N(\mu)}{X}\right) \sum_{\Pi} c_{\mu}(\Pi),$$

where $g \in C_0^{\infty}(\mathbb{R}^+)$. One can think of this sum as an average of L-functions.

The sum μ is over integral ideals of \mathbf{K} . Using the trace formula on (1.1) would lead to studying the analytic properties of the standard L-function of Π . Presumably, this should follow without too much difficulty from imitating the procedure in [S] for a standard L-function of forms over \mathbb{Q} . There the result is

$$\frac{1}{X} \sum_n g(n/X) \sum_{\pi} c_n(\pi) = O(X^{-A})$$

for any integer $A > 0$. This is equivalent to $L(s, \pi) = \sum_{n=1}^{\infty} \frac{c_n(\pi)}{n^s}$ being entire. One should be able to get the same bound for (1.1).

The Beyond Endoscopic approach to the Asai L-function would be to average (1.1) not over integral ideals but rational integers. Before stating the main theorem, we ask what do we expect from such a calculation? Our calculation should, for large X , behave as

$$(1.2) \quad \frac{1}{X} \sum_n g\left(\frac{n}{X}\right) \sum_{\Pi} c_n(\Pi) \approx \sum_{\Pi} \text{Res}_{s=1} L(s, \Pi, \text{Asai}) + O(X^{-\delta}), \delta > 0.$$

As we said, if the Asai L-function has a pole at $s = 1$, then Π is a base change from a form over \mathbb{Q} , specifically a form π of level D with nebentypus χ_D , see [A]. Lets label these forms by $B(D, \chi_D)$. So our heuristic in (1.2) becomes

$$(1.3) \quad \sum_{\Pi} \text{Res}_{s=1} L(s, \Pi, \text{Asai}) \approx \sum_{\pi \in B(D, \chi_D)} A(\pi) L(1, \text{sym}^2(\pi)) + O(X^{-\delta}), \delta > 0,$$

where $A(\pi)$ is a certain constant associated to π . This heuristic is still not quite accurate as (1.3) says every $\pi \in B(D, \chi_D)$ has a cuspidal base change. This is not true as Maass' cuspidal theta forms or dihedral forms constructed from Hecke characters over a quadratic field have base changes that are Eisenstein series. Say a theta form θ_{ω} is constructed from a Hecke character ω , then the associated Asai L-function decomposes as

$$L(s, \theta_{\omega}, \text{Asai}) = L(s, \chi_D)^2 L(s, \theta_{\omega}^2).$$

Thus our almost complete heuristic for the calculation is,

$$(1.4) \quad \sum_{\Pi} \text{Res}_{s=1} L(s, \Pi, \text{Asai}) \approx \sum_{\substack{\pi \in B(D, \chi_D) \\ \pi \neq \theta_{\omega}}} A(\pi) L(1, \text{sym}^2(\pi)) + O(X^{-\delta}), \delta > 0.$$

The heuristic in some sense is lacking as it only tells us information at or near $s = 1$. From the definition of the Asai L-function, we only get analyticity (ignoring whether there is a pole or not at $s = 1$) of the L-function up to the zeroes of $\zeta(2s)$. To remedy this we do an extra averaging in $\mathfrak{m} \in \mathbb{Z}$, which should, and does, remove the poles created by the zeroes of the $\zeta(2s)$.

Final heuristic!

$$(1.5) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g\left(\frac{\mathfrak{m}^2 n}{X}\right) \sum_{\Pi} c_n(\Pi) \approx \sum_{\substack{\pi \in B(D, \chi_D) \\ \pi \neq \theta_\omega}} A(\pi) L(1, \text{sym}^2(\pi)) + O(X^{-M}),$$

for any positive integer $M > 0$.

Our main result is:

Theorem 1.1. *Let $V = V_1 \times V_2 \in C_0^\infty(\mathbb{R}^+)^2$, and $g \in C_0^\infty(\mathbb{R}^+)$, such that $\int_0^\infty g(x) dx = 1$. $h(V, y)$ is a Bessel transform of V with index y defined in the next section. For any positive integer $M \geq 0$, and quadratic integer l with Galois conjugate l' and $(l, D) = 1$,*

(1) *{Cuspidal contribution}*

$$\begin{aligned} & \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) \overline{c_l(\Pi)} = \\ & 2\pi \left(1 + \frac{1}{D}\right) \left(\sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{\phi_{t,D} \neq \theta_{\omega_\mu}} h(V, t_\phi) a_{\frac{l'}{r^2}}(\phi_{t,D}) \overline{a_1(\phi_{t,D})} + \sum_{\phi_{k,D}} h(V, k_\phi) a_{\frac{l'}{r^2}}(\phi_k) \overline{a_1(\phi_k)} \right) + O(X^{-M}). \end{aligned}$$

(2) *{CSC:=Continuous spectrum contribution}*

Let $\mu_k = \frac{k\pi}{\log \epsilon_0}$, where ϵ_0 is the fundamental unit of \mathbf{K} , and $\tau_{it}(n) = \sum_{ab=n} \chi_D(a) \left(\frac{a}{b}\right)^{it}, \psi_{\mu_k}(y) := \sum_{\mathbb{N}(q)=y} \omega_{\mu_k}(q)$, where $\omega_{\mu_k}(x) = \left|\frac{x}{x'}\right|^{i\mu_k}$, then

$$\begin{aligned} & \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) CSC_{n,l} = \\ & 2\pi \left(1 + \frac{1}{D}\right) \left(\sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{k \in \mathbb{Z}} \frac{1}{\langle \psi_{\mu_k}, \psi_{\mu_k} \rangle} \frac{4\pi h(V_1, \mu_k) h(V_2, \mu_k) \psi_{\mu_k}\left(\frac{l'}{r^2}\right) \overline{\psi_{\mu_k}(1)}}{\cosh(\pi \mu_k) L(1, \chi_D) L(1, \omega_{\mu_k}^2)} + \right. \\ & \quad \left. \sum_{\substack{r \in \mathbb{N} \\ r|l}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(V_1, t) h(V_2, t) \tau_{it}\left(\frac{l'}{r^2}\right) \overline{\tau_{it}(1)}}{|L(1 - 2it, \chi_D)|^2} dt \right) + O(X^{-M}). \end{aligned}$$

We state a hypothesis needed for matching.

Hypothesis 1.2. *Assume for any Π , there exists $M_1, M_2 > 0$, such that*

$$(1.6) \quad \frac{1}{X} \sum_{n, \mathfrak{m}} g(\mathfrak{m}^2 n / X) c_n(\Pi) \ll (1 + |t_{\Pi_1}|)^{M_1} (1 + |t_{\Pi_2}|)^{M_2}.$$

This is an assumption we need to reduce the sum over Π from infinite to finite dimensional. This assumption follows easily from the functional equation, but one would like to obtain all information about the L-function using only the trace formula. It is still certainly conceivable

to get such a result purely from trace formula, but for this paper we make this uniformity assumption.

In this paper we show the following corollaries of the theorem, assuming Hypothesis 1.2.

Corollary 1.3. *If the Asai L-function associated to a representation Π has a pole at $s = 1$, then there exists a modular or Maass form ϕ such that $c_l(\Pi) = \sum_{\substack{r \in \mathbb{N} \\ r|l}} a_{\frac{l}{r^2}}(\phi)$.*

We define such a form as $\Pi := BC_{\mathbf{K}/\mathbb{Q}}(\phi)$.

Corollary 1.4. *$\zeta(2s)L(s, \Pi, \text{Asai})$ has analytic continuation to the entire plane with the exception of a simple pole at $s = 1$, where then $\Pi = BC_{\mathbf{K}/\mathbb{Q}}(\phi)$, for ϕ a form of level D and central character χ_D .*

2. DETAILS OF PAPER

There have been several papers using the trace formula to prove quadratic base change. These include [La], [Sa], and [Y]. The first two references [La], [Sa] proved base change by comparing a trace formula over the ground field (in our case \mathbb{Q}) with a certain "twisted" trace formula over the quadratic field. The comparison is made through an equality of the associated test functions used for each trace formula. The last reference [Y] is the most similar to our approach as it uses the relative trace formula. The Kuznetsov trace formula, which we use, is a special case of the relative trace formula, see [KL] for a derivation. However, there is still a comparison of trace formulas involved with [Y].

Naively, the beyond endoscopy idea is to start with one trace formula and extra averaging over spectral data (fourier coefficients, whittaker functions,...) and by "brute force" compute the answer. The same test function is used from start to end. To do such a calculation, one relies heavily upon analytic number theory techniques, not used in previous trace formula comparison papers. With these techniques, one gets a very good feel for the geometric side of the trace formula, and how one literally "builds" the spectral sum of the forms over \mathbb{Q} .

It seems a nice gift that the analytic continuation of the Asai L-function comes with the comparison. In the sense of getting analytic information on L-functions using a trace formula, the author is reminded of the beautiful paper of Jacquet and Zagier [JZ], which gets analytic continuation of the symmetric square L-function associated to a Maass form.

To the specifics of the paper, in section 3 we introduce the Kuznetsov trace formula stated in [BMP1]. They stated it for a general real number field, we only use it for a quadratic extension of \mathbb{Q} . In section 4, we describe what is the residue of the Asai L-function when it has a pole.

In section 5, we prove a crucial bijection between solutions of two different sets of equations. This bijection gets rid of the difficult to handle $e(\frac{\bar{x}}{c})$ when one "opens" the Kloosterman sum on the geometric side of the trace formula. With the bijection, in section 6, we implement it into the trace formula. Then, the hard analytic number theoretic computations are done in section 7. The problem is that all of our computations are done over the quadratic field $\mathbb{Q}(\sqrt{D})$, when our aim is to capture forms over \mathbb{Q} . In section 8 we use a formula of Zagier, [Z] which in its simplest form is, for $D|n$,

$$\sum_{\substack{r \in \mathcal{O}_{\mathbf{K}} / (\frac{n}{\sqrt{D}}) \\ rr' \equiv 1(n)}} e\left(\frac{r+r'}{n}\right) = \frac{1}{\sqrt{D}} S_D(1, 1, n).$$

Here $S_D(1, 1, n) = \sum_{a(n)^*} \chi_D(a) e\left(\frac{x+\bar{x}}{n}\right)$. This is our "bridge" to the trace formula over \mathbb{Q} .

In section 9, we deal with the continuous spectrum over $\mathbb{Q}(\sqrt{D})$. In section 10 we realize that the computation done in section 7 really is the geometric side of the trace formula over \mathbb{Q} . This requires an important theorem on the convolution of Bessel transforms in [H]. In order to get an equality or comparison of just cusp forms from $\mathbb{Q}(\sqrt{D})$ to \mathbb{Q} , we compare Fourier coefficients for the continuous spectrum over the 2 different fields. Then the continuous spectrum can be removed from both sides. Lastly, in section 11 we exploit the Hecke algebra associated with the problem, and match associated forms. As well, we show the analytic continuation of the Asai L-function.

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3. PRELIMINARIES

Let $\mathbf{K} = \mathbb{Q}(\sqrt{D})$, with D prime and assume it is of class number one. This is to simplify already intensive calculations. The ring of integers will be denoted $\mathcal{O}_{\mathbf{K}}$. Here the discriminant $D_{\mathbf{K}} = D$, and the different is generated by $\delta = \sqrt{D}$. Likewise, define the absolute norm of an ideal c as $\mathbb{N}(c)$, and the norm of an element $z \in \mathcal{O}_{\mathbf{K}}$ as $\mathbb{N}(z)$. We denote the non-trivial automorphism in this field by $x \rightarrow x'$. We use the standard notation for the exponential $e(\text{Tr}_{\mathbf{K}/\mathbb{Q}}(x)) := \exp(2\pi i(\frac{x}{\delta} + \frac{x'}{\delta'}))$.

A bit of terminology is needed before we can define the Kuznetsov trace formula. We closely follow, and leave much of the details, to [BMP1]. We consider the algebraic group $\mathbf{G} = R_{K/\mathbb{Q}}(\text{SL}_2)$ over \mathbb{Q} obtained by restriction of scalars applied to SL_2 over K . We have

$$(3.1) \quad G := \mathbf{G}_{\mathbb{R}} \cong \text{SL}_2(\mathbb{R})^2, \quad \mathbf{G}_{\mathbb{Q}} \cong \{(x, x') : x \in \text{SL}_2(K)\},$$

G contains $K := \text{SO}_2(\mathbb{R})^2$ as a maximal compact subgroup.

The image of $\text{SL}_2(\mathcal{O}) \subset \text{SL}_2(F)$ corresponds to $\mathbf{G}_{\mathbb{Z}}$. This is a discrete subgroup of $\mathbf{G}_{\mathbb{R}}$ with finite covolume. It is called the *Hilbert modular group*. We label it Γ .

In this paper, we are concerned with functions on $\Gamma \backslash G$. We restrict ourselves to *even functions*: $f(-g) = f(g)$. By $L^2(\Gamma \backslash G)^+$ we mean the Hilbert space of (classes of) even functions that are left invariant under Γ , and square integrable on $\Gamma \backslash G$ for the measure induced by the Haar measure. This Hilbert space contains the closed subspace $L_c^2(\Gamma \backslash G)^+$ generated by integrals of Eisenstein series. The orthogonal complement $L_d^2(\Gamma \backslash G)^+$ of $L_c^2(\Gamma \backslash G)^+$ is the closure of $\sum_{\Pi} V_{\Pi}$, where Π runs through an orthogonal family of closed irreducible subspaces for the G -action in $L^2(\Gamma \backslash G)$ by right translation.

3.1. Irreducible unitary representations. Each representation Π has the form $\Pi = \Pi_1 \otimes \Pi_2$, with Π_j an even unitary irreducible representation of $\text{SL}_2(\mathbb{R})$. Table 1. of [BMP1] lists the possible isomorphism classes for each Π_j . For each Π we define a spectral parameter $\nu_{\Pi} = (\nu_{\Pi,1}, \nu_{\Pi,2})$, with $\nu_{\Pi,j}$ as in the last column of the corresponding table. There are Casimir operators C_j acting on each coordinate for $1 \leq j \leq 2$. The eigenvalue $\lambda_{\Pi} \in \mathbb{R}^2$ is given by

$\lambda_{\Pi,j} = \frac{1}{4} - \nu_{\Pi,j}^2$. We note that if Π_j lies in the complementary series, $\lambda_{\Pi,j} \in (0, \frac{1}{4})$ and if Π_j is isomorphic to a discrete series representation D_b^\pm , $b \in 2\mathbb{Z}$, $b \geq 2$, then $\lambda_{\Pi,j} = \frac{b}{2}(1 - \frac{b}{2}) \in \mathbb{Z}_{\leq 0}$.

The constant functions give rise to $\Pi = \mathbf{1} := \otimes_j \mathbf{1}$. It occurs with multiplicity one. If V_Π does not consist of the constant functions, then $\Pi_j \neq 1$ for all j .

3.2. Automorphic forms and Fourier coefficients. The elements of V_Π that transform on the right according to a character of the maximal compact subgroup K are square integrable automorphic forms. The r -th Fourier coefficients of the automorphic form f_Π at the cusp κ , is $a_{r,\kappa}(f_\Pi)$ where κ is the defined as in [BMP1]. They are essentially independent of the choice of automorphic form chosen in V_Π , and so we call them $a_{r,\kappa}(\Pi)$. For the purposes of the Kuznetsov trace formula we define

$$(3.2) \quad d_\kappa(r, \nu) := \frac{1}{\sqrt{D}} \prod_{j=1}^2 \frac{(-1)^{q(\nu_j)/2} (2\pi|r_j|)^{-1/2}}{\Gamma(\frac{1}{2} + \nu_j + \frac{1}{2}q(\nu_j)\text{sign}(r_j))},$$

where $q(\nu_j) = 0$ if Π_j is the principal or complementary series, and equals b if Π_j is the discrete series with minimal weight b .

Then the coefficients associated to the trace formula are defined as $c_r(\Pi) := \frac{a_{r,\infty}(\Pi)}{d_\infty(r, \nu)}$. This is very similar to Equation (17) of [BMP1]. We do not indicate the cusp in the definition of $d_\kappa(r, \nu)$ as we are in class number one, and so we have to only consider the cusp ∞ .

The Kuznetsov trace formula then is

$$(3.3) \quad \sum_{\Pi \neq \mathbf{1}} h(V, \nu_\Pi) c_\mu(\Pi) \overline{c_\nu(\Pi)} + \{CSC_{\mu,\nu}\} = \\ = \sum_{c \neq 0}^{\infty} \frac{1}{\mathbb{N}(c)} S(\nu, \mu, c) V(4\pi\sqrt{\mu\nu}/c),$$

where $\mu, \nu \in \mathcal{O}_K$. $V = V_1 \times V_2$ is a test function in $C_0^\infty(\mathbb{R}^+)^2$. The transforms associated to the archimedean parameter ν_Π are

$$(3.4) \quad h(V_i, \nu_{\Pi_i}) = \begin{cases} i^k \int_0^\infty V_i(x) J_{\nu_{\Pi_i}-1}(x) x^{-1} dx & \text{if } \nu_{\Pi_i} \in 2\mathbb{Z}; \\ \int_0^\infty V_i(x) B_{2\nu_{\Pi_i}}(x) x^{-1} dx & \text{if } \nu_{\Pi_i} \in i\mathbb{R}. \end{cases}$$

Here, $B_{2it}(x) = (2 \sin(\pi it))^{-1} (J_{-2it}(x) - J_{2it}(x))$, where $J_\mu(x)$ is the standard J -Bessel function of index μ . The Kloosterman sum

$$(3.5) \quad S(r, s, c) := \sum_{x \in (\mathcal{O}_K^*/c\mathcal{O}_K^*)} e(Tr_{K/\mathbf{Q}}(\frac{r\bar{x} + sx}{c})),$$

where $x\bar{x} \equiv 1(c)$. We do not describe the continuous spectrum here as it is quite lengthy, but elaborate on it greatly in Section 9. Other descriptions and applications of the Kuznetsov trace formula for number fields include [BMP1], [BMP2], [KL], and [V].

4. RESIDUE OF THE ASAI L-FUNCTION

We answer the question of what is the residue of the Asai L-function for a lifted holomorphic Hilbert modular form F of parallel weight k . The other spectral cases are similar. As stated earlier, the Asai L-function of F has a pole at $s = 1$ if the form is a base change

$F = BC_{\mathbb{Q}(\sqrt{D})/\mathbb{Q}}(f)$ for a form f of level D and central character χ_D . Following [A], let $\mathbb{H}_2 := \mathbb{H} \times \mathbb{H}$, then for $F(w_1, w_2)$,

$$\langle F, F, SL_2(\mathcal{O}) \rangle = \int_{SL_2(\mathcal{O}) \backslash \mathbb{H}_2} |\mathbb{N}(y)|^k |F(w_1, w_2)|^2 d^x w,$$

where $w_j = x_j + y_j i, j = 1, 2$. Similarly, for $f \in S_0(\Gamma_0(D), \chi_d)$, we have the standard inner product

$$\langle f, f, \Gamma_0(D) \rangle = \int_{\Gamma_0(D) \backslash \mathbb{H}} |f(z)|^2 y^{-2} dx dy.$$

Then the residue of the Asai L-function is

$$\text{Res}_{s=1} L(s, F, \text{Asai}) = \frac{(4\pi)^k}{\Gamma(k)} \frac{\langle F, F, SL_2(\mathcal{O}) \rangle}{\zeta(2) \langle f, f, \Gamma_0(D) \rangle}.$$

Now including that we take an orthonormal basis of F and the Fourier coefficient normalization of (3.2), we get

$$(4.1) \quad \frac{1}{\langle F, F, SL_2(\mathcal{O}) \rangle} \text{Res}_{s=1} \sum_{n=1}^{\infty} c_n(F) n^{-s} = \frac{12D}{\pi} \frac{1}{\langle f, f, \Gamma_0(D) \rangle}.$$

Taking into consideration we have an extra average over \mathfrak{m} in the Theorem removing the $\zeta(2)$, and that the sum of forms f over \mathbb{Q} are an orthonormal basis for $\Gamma_0(D)/\mathbb{H}$ with norm

$$\|f\|_2 := \frac{1}{D(1 + \frac{1}{D})} \langle f, f, \Gamma_0(D) \rangle,$$

we have for unnormalized Fourier coefficients $a_n(\Pi)$,

$$(4.2) \quad \text{Res}_{s=1} \sum_{n=1}^{\infty} a_n(F) n^{-s} = \frac{2\pi}{\|f\|_2} = 2\pi(1 + \frac{1}{D}).$$

5. NUMBER-THEORETIC LEMMAS

We prove some number-theoretic lemmas that are crucial to our calculation.

Definition 5.1. Let $\overline{X}(c, n)$ denote the equivalence classes (x, m) with $x \in \mathcal{O}_{\mathbf{K}}$ such that $(x, c) = 1$, $m \in \mathbb{Z}$, and

$$\delta' c' x + \delta c x' = n - m \mathbf{N}(\delta c).$$

Here we say that x is equivalent to y if $x \equiv y \pmod{c}$. Let $X(c, n)$ be a set of representatives for the classes in $\overline{X}(c, n)$.

Proposition 5.2. Let $\gamma = (-1)^e \eta^m$ where η is the fundamental unit of the field \mathbf{K} and $e \in \{0, 1\}, m \in \mathbb{Z}$. Let $(x, m) \in X(c, 0)$, then $c = \gamma a$ or $\gamma b \sqrt{D}$, $a, b \in \mathbb{N}$. Further, $\eta^m \overline{x} \equiv \eta^m \overline{x'}(\delta a)$, if $c = \gamma a$ and $\eta^m \overline{x} \equiv -\eta^m \overline{x'}(Db)$, if $c = \gamma b \sqrt{D}$. Also we can choose $\eta^m \overline{x} \equiv k + h \sqrt{D}(\delta a)$, $k, h \in \mathbb{Z}$ with $2h \equiv 0(a)$ if $c = \gamma a$. Likewise, $\eta^m \overline{x} \equiv k + h \sqrt{D}(Db)$, $k, h \in \mathbb{Z}$ with $2k \equiv 0(Db)$ if $c = \gamma b \sqrt{D}$.

Proof. It is clear that $(x, m) \in X(c, 0)$ implies $c \equiv 0(c')$, likewise $c' \equiv 0(c)$. This implies by [FT](V.1.16) $c = \eta^m a \in Z$ or $c = \eta^m b \sqrt{D}$, $a, b \in \mathbb{Z}$. Without loss of generality take $c = \gamma a$, then this implies $\delta a(\eta^m x - \eta^m x') \equiv 0(Da^2)$, or $\eta^m x \equiv \eta^m x'(\delta a)$. This implies $\eta^m \overline{x} \equiv \eta^m \overline{x'}(\delta a)$. The identical calculation for $c = \gamma b \sqrt{D}$, gives $\eta^m \overline{x} = -\eta^m \overline{x'}(Db)$. The last statements are clear if we let $\eta^m x = k + l \sqrt{D}$. □

It is assumed, unless stated otherwise, $n \neq 0$.

Proposition 5.3. *Let $(x, m) \in X(c, n)$. Let $\bar{x} \in O_{\mathbf{K}}$ be an inverse of x modulo c . Then there exists an r such that $rr' \equiv 1 \pmod{n}$ and*

$$(5.1) \quad \bar{x} = \frac{\delta c r + \delta' c'}{n},$$

The r is uniquely determined modulo $\frac{n}{\delta}$ by the equivalence class of x , and the map from $X(c, n)$ to the set r modulo n is injective.

Proof. Set

$$r = \frac{n\bar{x} - \delta' c}{\delta c}.$$

Note that r is an integer in the field \mathbf{K} because

$$n\bar{x} - \delta' c' = (\delta' c' x + \delta c x')\bar{x} - \delta' c' = \delta' c' (x\bar{x} - 1) + \delta c x' \bar{x} \equiv 0 \pmod{\delta c}$$

It is clear that r is determined by \bar{x} . If we replace \bar{x} by $\bar{y} = \bar{x} + \mu c$, r is replaced by

$$(5.2) \quad s = r + \mu \frac{n}{\sqrt{D}}$$

If x and y in $X(c, n)$ are both associated to r , then $\bar{x} = \bar{y}$. Therefore $x \equiv y \pmod{c}$. Finally,

$$rr' = \left(\frac{n\bar{x} - \delta' c'}{\delta c} \right) \left(\frac{n\bar{x}' - \delta c}{\delta' c'} \right) = 1 + \frac{n^2 \bar{x} \bar{x}' - n\bar{x} \delta c - n\bar{y} \delta' c'}{\mathbf{N}(\delta c)} = 1 + n \frac{n\bar{x} \bar{x}' - \bar{x} \delta c - \bar{x}' \delta' c'}{\mathbf{N}(\delta c)}.$$

But

$$n\bar{x} \bar{x}' = (\delta' c x + \delta c x') \bar{x} \bar{x}' = \delta' c' x \bar{x} \bar{x}' + \delta c \bar{x} x' \bar{x}'$$

so we have

$$\begin{aligned} rr' &= 1 + n \frac{\delta' c' x \bar{x} \bar{x}' + \delta c \bar{x} x' \bar{x}' - \bar{x} \delta c - \bar{x}' \delta' c'}{\mathbf{N}(\delta c)} \\ &= 1 + n \left[\frac{\delta' c' (x\bar{x} - 1) \bar{x}' + \delta c (x' \bar{x}' - 1) \bar{x}}{\mathbf{N}(\delta c)} \right] \\ &= 1 + n \left[\frac{\bar{x}' q}{\delta} + \frac{\bar{x} q'}{\delta'} \right], q \in \mathbf{O}_{\mathbf{K}}. \end{aligned}$$

The expression in brackets is an integer of the field, so $rr' \equiv 1 \pmod{n}$. It is also easy to check that from (5.2),

$$(r + \mu \frac{n}{\sqrt{D}})(r' - \mu' \frac{n}{\sqrt{D}}) \equiv rr' \equiv 1(n).$$

□

Definition 5.4. *Let c be an integer in $O_{\mathbf{K}}$. Set $d = (c, c')$. Assume that $d|n$. Let $Y(c, n)$ be the set of classes $r \in (O_{\mathbf{K}}/(\frac{n}{\delta}))^*$ such that*

- (a) $(\delta c/d)r + (\delta' c'/d) \equiv 0 \pmod{\frac{n}{d}}$
- (b) $(\delta c/d)r + (\delta' c'/d) \not\equiv 0 \pmod{\frac{n}{k}}$ if $k|d$ and $k < d$.

It is easy to check this definition is well-defined on classes $r \in (O_{\mathbf{K}}/(\frac{n}{\delta}))^*$.

Proposition 5.5. *The map $i : (x, m) \rightarrow r$ defines a bijection between $X(c, n)$ and $Y(c, n)$.*

Proof. Let $(x, m) \in X(c, n)$. We show that the associated r belongs to $Y(c, n)$. Then

$$\bar{x} = \frac{\delta cr + \delta' c'}{n} = \frac{(\delta c/d)r + (\delta' c'/d)}{(n/d)}$$

Therefore, $\frac{\delta cr}{d} + \frac{\delta' c'}{d} \equiv 0 \pmod{\frac{n}{d}}$ and (a) is satisfied. Suppose that m is a proper divisor of d and let $k = d/m$. We claim that $\frac{\delta c}{d}r + \frac{\delta' c'}{d} \not\equiv 0 \pmod{\frac{n}{k}}$. If this were not the case, we would have

$$\bar{x} = \frac{(\delta' c'/d) + (\delta c/d)r}{(n/d)} = m \frac{(\delta' c'/d) + (\delta c/d)r}{(n/k)}$$

This would imply that m divides \bar{x} , which contradicts the fact that \bar{x} is a unit modulo c . Therefore (b) is satisfied and $r \in Y(c, n)$. Furthermore, the map i is injective on $X(c, n)$ by Proposition 5.3. Next, assume that $Y(c, n)$ is non-empty. Let $r \in O_{\mathbf{K}}$ be relatively prime to n and assume that $r \pmod{n}$ belongs to $Y(c, n)$. Set

$$(5.3) \quad \xi = \frac{(c/d)r - (c'/d)}{n/\delta d} = \frac{c' - cr}{n/\delta}$$

Then ξ is relatively prime to d because $(c/d)r - (c'/d) \not\equiv 0 \pmod{n/k\delta}$ for all proper divisors k of d . On the other hand, if q is a common factor of both ξ and c/d , then $q|c'/d$. But $(c/d, c'/d) = 1$ so q is a unit. This proves that ξ is prime to both d and c/d , and hence is a unit modulo c . Now choose $x \in O_{\mathbf{K}}$ such that $x\xi \equiv 1 \pmod{c}$ and set $\bar{x} = \xi$. Then

$$x\bar{x} = 1 + \mu c$$

for some $\mu \in O_{\mathbf{K}}$.

We claim that there exists a $m \in \mathbb{Z}$ such that

$$\delta' c' x + \delta c x' = n - m\mathbf{N}(\delta c).$$

We notice first

$$\frac{n - \delta' c' x}{\delta c} \in O_{\mathbf{K}}.$$

Indeed, by (5.3) we have $\delta' c' = \bar{x}n - \delta cr$, so

$$\frac{n - \delta' c' x}{\delta c} = \frac{n - (\bar{x}n - \delta cr)x}{\delta c} = \frac{n(1 - x\bar{x}) + \delta crx}{\delta c} = rx + \frac{n\mu}{\delta} \in O_{\mathbf{K}}.$$

Now, by the above argument, there exists some $m \in O_{\mathbf{K}}$ such that

$$x' + m\delta' c' = \frac{n - \delta' c' x}{\delta c}.$$

But this implies

$$\delta c x' + \delta' c' x = n - m\mathbf{N}(\delta c),$$

which implies $m \in \mathbb{Z}$. Now if we take

$$s = r + \gamma \frac{n}{\delta},$$

its clear (5.3) changes $\xi \rightarrow \xi + \gamma c$, and the rest of the argument follows analogously. Therefore equivalence classes map to equivalence classes. This proves the surjectivity and hence the bijection. \square

6. TAKING GEOMETRIC SIDE OF TRACE FORMULA

In this section, we rewrite the left hand side of Theorem 1.1 and discuss what the next sections will involve. Using the Kuznetsov formula the left hand side of Theorem 1.1 equals,

$$(6.1) \quad (L) := \frac{1}{X} \sum_{\mathfrak{m}, n \in \mathbb{Z}} g(\mathfrak{m}^2 n / X) \left(\sum_{\Pi \neq 1} h(V, \nu_{\Pi}) c_n(\Pi) \overline{c_l(\Pi)} + \{CSC_{n,l}\} \right) =$$

$$\frac{1}{X} \sum_{\mathfrak{m}, n \in \mathbb{Z}} g(\mathfrak{m}^2 n / X) \sum_c \frac{1}{\mathbb{N}(c)} S(n, l, c) V_1\left(\frac{4\pi\sqrt{nl}}{c}\right) V_2\left(\frac{4\pi\sqrt{nl'}}{c'}\right).$$

We now break up the Kloosterman sums and gather all the n -terms. We can do this because c and n sum are finite due to the support of g and V .

$$(6.2) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_c \frac{1}{\mathbb{N}(c)} \sum_{x(c)^*} e\left(\frac{\bar{x}l}{\delta c} + \frac{\bar{x}'l'}{\delta' c'}\right)$$

$$\left\{ \sum_{n \in \mathbb{Z}} e\left(n\left(\frac{x}{\delta c} + \frac{x'}{\delta' c'}\right)\right) g(\mathfrak{m}^2 n / X) V_1\left(\frac{4\pi\sqrt{nl}}{c}\right) V_2\left(\frac{4\pi\sqrt{nl'}}{c'}\right) \right\},$$

where \bar{x} is the multiplicative inverse of $x(c)$.

Since the term in brackets is smooth, we can and do apply Poisson summation to the n -sum as well as a change of variables $t \rightarrow Xt$:

$$(6.3) \quad \sum_{\mathfrak{m}} \sum_c \frac{1}{\mathbb{N}(c)} \sum_{x(c)^*} e\left(\frac{\bar{x}l}{\delta c} + \frac{\bar{x}'l'}{\delta' c'}\right)$$

$$\left\{ \sum_m \int_{-\infty}^{\infty} e\left(Xt\left(\frac{x\delta'c' + x'c\delta - \mathbb{N}(c\delta)m}{\mathbb{N}(c\delta)}\right)\right) g(\mathfrak{m}^2 t) V_1\left(\frac{4\pi\sqrt{lXt}}{c}\right) V_2\left(\frac{4\pi\sqrt{l'Xt}}{c'}\right) dt \right\}.$$

We hope the confusion of using \mathfrak{m} - and m -sums is not too big, the use of one will go away shortly.

As we have fixed l , let

$$I_{\mathfrak{m}}(n, c, X) := \int_{-\infty}^{\infty} e\left(\frac{Xtn}{\mathbb{N}(\delta c)}\right) g(\mathfrak{m}^2 t) V_1\left(\frac{4\pi\sqrt{Xtl}}{c}\right) V_2\left(\frac{4\pi\sqrt{Xtl'}}{c'}\right) dt$$

Then (L) is equal to

$$\sum_{\mathfrak{m}} \sum_c \frac{1}{\mathbb{N}(c)} \sum_{x(c)^*} e\left(\frac{\bar{x}l}{\delta c} + \frac{\bar{x}'l'}{\delta' c'}\right) \sum_{m \in \mathbb{Z}} I_{\mathfrak{m}}(x\delta'c' + x'c\delta - \mathbb{N}(c\delta)m, c, X)$$

Let $X'(c, n)$ be the set of solutions (x, m) of the equation

$$\delta'c'x + \delta cx' - \mathbb{N}(c\delta)m = n$$

where x range over a fixed set of representatives of $(O_{\mathbf{K}}/c)^*$ and $m \in \mathbb{Z}$. Then (L) is equal to

$$\sum_{\mathfrak{m}} \sum_{n \in \mathbb{Z}} \sum_c \frac{1}{\mathbb{N}(c)} \sum_{(x, m) \in X'(c, n)} e\left(\frac{\bar{x}l}{\delta c} + \frac{\bar{x}'l'}{\delta' c'}\right) I_{\mathfrak{m}}(n, c, X)$$

Note there is a bijection between the set $(x, m) \in X'(c, n)$ and the set of equivalence classes (x, m) in $X(c, n)$ from Definition 5.1. Thus we may replace the sum over $X'(c, n)$ with a sum over $X(c, n)$:

$$(L) = \sum_{\mathfrak{m}} \sum_{n \in \mathbf{Z}} \sum_c \frac{1}{\mathbb{N}(c)} \sum_{x \in X(c, n)} e\left(\frac{\bar{x}l}{\delta c} + \frac{\bar{x}'l'}{\delta'c'}\right) I_{\mathfrak{m}}(n, c, X)$$

Finally, let

$$(6.4) \quad A_{n,X} := \sum_{\mathfrak{m}} \sum_c \frac{1}{\mathbb{N}(c)} \sum_{x \in X(c, n)} e\left(\frac{\bar{x}l}{\delta c} + \frac{\bar{x}'l'}{\delta'c'}\right) I_{\mathfrak{m}}(n, c, X);$$

then

$$(6.5) \quad (L) = \sum_{n \in \mathbf{Z}} A_{n,X}$$

Let us fix the generator of the different as \sqrt{D} .

Lemma 6.1. *For n defined above, $D|n$.*

Proof. Let $c := \alpha + \beta\sqrt{D}$, $x := w + z\sqrt{D}$, then

$$(6.6) \quad n = (-\sqrt{D})(\alpha - \beta\sqrt{D})(w + z\sqrt{D}) + (\sqrt{D})(\alpha + \beta\sqrt{D})(w - z\sqrt{D}) + Dm(\alpha^2 - D\beta^2).$$

This equals

$$-2D(\alpha z - \beta w) + Dm(\alpha^2 - 2\beta^2).$$

□

We know see that (6.5) equals,

$$(6.7) \quad \sum_{D|n \in \mathbf{Z}} A_{n,X}.$$

Now for $n \neq 0$, we can use the bijection of Proposition 5.5 to rewrite $A_{n,X}$ as a sum over $r \in Y(c, n)$:

$$A_{n,X} = \sum_{\mathfrak{m}} \sum_{\substack{r \in O_{\mathbf{K}}/(\frac{n}{\delta}) \\ rr' \equiv 1(n)}} e\left(\frac{rl + r'l'}{n}\right) \sum_{\substack{c \\ r \in Y(c, n)}} \frac{1}{\mathbb{N}(c)} e\left(\frac{-1}{n} \left(\frac{lc'}{c} + \frac{l'c}{c'}\right)\right) I_{\mathfrak{m}}(n, c, X)$$

Definition 6.2. *Let $X_n(r)$ be the set c such that $r \in Y(c, n)$.*

Definition 6.3. *Let $H_{n,\mathfrak{m}}(x, x') := \frac{1}{xx'} e\left(\frac{x'l'}{-nx'} + \frac{x'l}{-nx}\right) I_{\mathfrak{m}}(n, x, 1)$.*

The main result of the calculations can be broken down into the cases: $n = 0$, and $n \neq 0$.

We let

$$\delta(l, l') = \begin{cases} 1 & \text{if } l = l'; \\ 0 & \text{if } l \neq l'. \end{cases}$$

In Section 7.1 we show for any integer $M \geq 0$,

$$(6.8) \quad A_{0,X} = \frac{\delta(l, l')(1 + \frac{1}{D})}{2} \int_0^\infty \int_0^\infty V_1(x) V_2(y) \frac{dx dy}{xy} + O(X^{-M}).$$

In Section 7.2 for fixed $n \neq 0$, $r \in (\mathbb{Z}/n)^*$, and any integer $M \geq 0$, we have

$$(6.9) \quad \sum_{\mathfrak{m}} \sum_{c \in X_n(r)} \frac{1}{\mathbb{N}(c)} e\left(\frac{-1}{n} \left(\frac{lc'}{c} + \frac{l'c}{c'}\right)\right) I_{\mathfrak{m}}(n, c, X) = \frac{(1 + \frac{1}{D})}{\sqrt{D}} \int_0^\infty \int_0^\infty H_n(x, y) dx dy + O(X^{-M}).$$

In Section 8 for $(l, D) = 1$, we get the r -sum in $A_{n,X}$ is replaced by a sum of Kloosterman sums twisted by the character χ_D attached to the quadratic field \mathbf{K} . Specifically this is

$$\sum_{\substack{r \in O_{\mathbf{K}}/(\frac{n}{\delta}) \\ rr' \equiv 1(n)}} e\left(\frac{rl + r'l'}{n}\right) = \frac{1}{\sqrt{D}} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D\left(\frac{ll'}{r^2}, 1, \frac{Da}{r}\right).$$

Combining sections 7.2 and 8 we get

$$A_{n,X} = \frac{(1 + \frac{1}{D})}{D} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D\left(\frac{ll'}{r^2}, 1, \frac{Da}{r}\right) \int_0^\infty \int_0^\infty H_n(x, y) dx dy + O(X^{-M}).$$

Lastly, sections 7.1, 7.2, and 8 then show

$$(6.10) \quad (L) = \left(1 + \frac{1}{D}\right) \frac{\delta(l, l')}{2} \int_0^\infty \int_0^\infty V_1(x) V_2(y) \frac{dx dy}{xy} + \frac{(1 + \frac{1}{D})}{D} \sum_{a=1}^\infty \frac{1}{Da} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D\left(\frac{ll'}{r^2}, 1, \frac{Da}{r}\right) \times \\ \int_0^\infty \int_0^\infty H_{Da}(x, y) dx dy + O(X^{-M}).$$

7. COMPUTING $A_{q,X}$.

We separate the cases of $A_{0,X}$, and $A_{q,X}$, $q \neq 0$. The former is computed first.

7.1. Evaluating $A_{0,X}$. We note from Proposition 5.2 the pair $(x, m) \in X(\eta^j a, 0)$, bijectively corresponds to $(\eta^j x, m) \in X(a, 0)$, where $j, a \in \mathbb{Z}$. Therefore, since we count $x \pmod{c}$ up to equivalence class in the trace formula it suffices to just look at a fixed j .

Using Proposition 5.2, we get from inside $A_{0,X}$ for $c = a$,

$$\sum_{x \in X(a, 0)} e\left(\frac{\eta^j \overline{x} l}{\delta a} + \frac{\eta^j \overline{x'} l'}{\delta' a}\right) = \sum_{\substack{y(a)^* \\ y \in \mathbb{Z}}} e\left(\frac{y(l - l')}{a}\right).$$

For $c = \sqrt{D}b$, we get

$$\sum_{x \in X(b\sqrt{D}, 0)} e\left(\frac{\eta^j \overline{x} l}{Db} + \frac{\eta^j \overline{x'} l'}{Db}\right) = \sum_{\substack{y(Db)^* \\ y \in \mathbb{Z}}} e\left(\frac{y(l - l')}{Db}\right).$$

Introducing these to $A_{0,X}$ gives,

$$(7.1) \quad A_{0,X} := \sum_{\mathfrak{m}} \left[\sum_{a=1}^{\infty} \frac{1}{a^2} \sum_{\substack{y(a)^* \\ y \in \mathbb{Z}}} e\left(\frac{y(l-l')}{a}\right) \int_{-\infty}^{\infty} g(\mathfrak{m}^2 t) V_1\left(\frac{4\pi\sqrt{X}lt}{a}\right) V_2\left(\frac{4\pi\sqrt{X}l't}{a}\right) dt + \right. \\ \left. \frac{1}{D} \sum_{b=1}^{\infty} \frac{1}{b^2} \sum_{\substack{y(Db)^* \\ y \in \mathbb{Z}}} e\left(\frac{y(l-l')}{\delta b}\right) \int_{-\infty}^{\infty} g(\mathfrak{m}^2 t) V_1\left(\frac{4\pi\sqrt{X}lt}{b\sqrt{D}}\right) V_2\left(\frac{4\pi\sqrt{X}l't}{b\sqrt{D}}\right) dt \right].$$

Proposition 7.1. *For any integer $M \geq 0$,*

$$(7.2) \quad A_{0,X} = \frac{\delta(l, l')(1 + \frac{1}{D})}{2} \int_0^{\infty} \int_0^{\infty} V_1(x) V_2(y) \frac{dx dy}{xy} + O(X^{-M}).$$

Proof. Let

$$H_{\mathfrak{m}}(x, y) := \frac{1}{xy} \int_0^{\infty} g(\mathfrak{m}^2 t) V_1\left(\frac{4\pi\sqrt{lt}}{x}\right) V_2\left(\frac{4\pi\sqrt{l't}}{y}\right) dt,$$

then notice

$$H_{\mathfrak{m}}(x, y) = H_1(\mathfrak{m}x, \mathfrak{m}y).$$

For economy we use $H(x, y)$ for $H_1(x, y)$.

We then have

$$(7.3) \quad A_{0,X} = \frac{1}{X} \sum_{\mathfrak{m}} \left[\sum_{a=1}^{\infty} \sum_{\substack{y(a)^* \\ y \in \mathbb{Z}}} e\left(\frac{y(l-l')}{\delta a}\right) H\left(\frac{\mathfrak{m}a}{\sqrt{X}}, \frac{\mathfrak{m}a}{\sqrt{X}}\right) + \right. \\ \left. \sum_{b=1}^{\infty} \sum_{\substack{y(Db)^* \\ y \in \mathbb{Z}}} e\left(\frac{y(l-l')}{\delta b}\right) H\left(\frac{\mathfrak{m}b\sqrt{D}}{\sqrt{X}}, \frac{\mathfrak{m}b\sqrt{D}}{\sqrt{X}}\right) \right].$$

It is sufficient to focus on the first part of (7.3).

Let $\hat{H}(s_1, s_2) := \int_0^{\infty} \int_0^{\infty} H(x, y) x^{s_1-1} y^{s_2-1} dx dy$. Then by Mellin inversion, the sum over equals

$$\sum_{\mathfrak{m}} \left(\frac{1}{2\pi i} \right)^2 \int_{(\sigma_1, \sigma_2)} \hat{H}(s_1, s_2) \left(\frac{\sqrt{X}}{\mathfrak{m}} \right)^{s_1} \left(\frac{\sqrt{X}}{\mathfrak{m}} \right)^{s_2} L(s_1 + s_2) ds_1 ds_2,$$

for $\sigma_i > 0$ sufficiently large. Here

$$L(s_1 + s_2) := \sum_{a=1}^{\infty} \frac{f_a\left(\frac{l-l'}{\delta}\right)}{a^{s_1+s_2}},$$

where $f_n(y) = \sum_{x(n)^*} e\left(\frac{xy}{n}\right)$ is the classical Ramanujan sum. Notice the exponential sum in $L(s)$ is independent of the choice of representatives as $\frac{l-l'}{\delta} \in \mathbb{Z}$. Also note we removed the indication that the residue classes modulo n are rational integer classes.

Using the fact $f_n(y) = \mu\left(\frac{n}{(y, n)}\right) \frac{\phi(n)}{\phi\left(\frac{n}{(y, n)}\right)}$ and multiplicativity,

$$(7.4) \quad L(s_1 + s_2) = \prod_{p \nmid (l-l')} \left(1 - \frac{1}{p^{s_1+s_2}}\right) \prod_{p \mid (l-l')} \left(1 - \frac{1}{p^{s_1+s_2}}\right) \left(1 + \frac{1}{p^{(s_1+s_2)-1}}\right) = \frac{\prod_{p \mid (l-l')} \left(1 + \frac{1}{p^{(s_1+s_2)-1}}\right)}{\zeta(2s)}.$$

If $l = l'$ then

$$L(s_1 + s_2) = \sum_{a=1}^{\infty} \frac{\phi(a)}{a^{s_1+s_2}} = \frac{\zeta((s_1 + s_2) - 1)}{\zeta(s_1 + s_2)}.$$

Since H is smooth and compactly supported $\hat{H}(s_1, s_2) \ll_N [(1 + |t_1|)(1 + |t_2|)]^{-N}$, and we can clearly interchange the integral and \mathfrak{m} -sum. Notice also the \mathfrak{m} sum inside of the integral is $\zeta(s_1 + s_2)$.

So we have after recollecting terms,

$$(7.5) \quad \sum_j \frac{1}{2\pi i X} \int_{(\sigma_1, \sigma_2)} \hat{H}(s_1, s_2) \left(\sqrt{X}\right)^{s_1+s_2} L(s_1 + s_2) \zeta(s_1 + s_2) ds_1 ds_2.$$

Lets assume $l = l'$, notice now how the poles of $L(s_1 + s_2)$ are at the poles of $\zeta(s_1 + s_2)$, however these are exactly removed by our extra summation over $\mathfrak{m} \in \mathbb{Z}!$

Now using contour integration $\sigma_i \rightarrow -M, M > 0, i = 1, 2$, we get

$$(7.6) \quad \frac{X}{2} \hat{H}(1, 1) + \left(\frac{1}{2\pi i X}\right)^2 \int_{(-M, -M)} \hat{H}(s_1, s_2) \left(\sqrt{X}\right)^{s_1+s_2} L(s_1 + s_2) \zeta(s_1 + s_2) ds_1 ds_2.$$

So using trivial bounds on the integral,

$$\frac{X}{2} \hat{H}(1, 1) + O(X^{-M}).$$

Similar analysis can be done for $l \neq l'$, here however we get no pole from the L-function (7.4) and the term here is $O(X^{-M})$.

For the b -sum we only note here

$$\sum_{b=1} \frac{\phi(Db)}{(Db)^{2s}} = \frac{D}{D^{2s}} \sum_{b=1} \frac{\phi(b)}{b^{2s}}.$$

The same argument in (7.1) gives $\frac{X}{2D} \hat{H}(1, 1) + O(X^{-M})$.

Therefore (7.1) equals

$$\frac{1}{X} \left(\frac{X}{2} \hat{H}(1, 1) + \frac{X}{2D} \hat{H}(1, 1) \right) + O(X^{-M}) = \frac{(1 + \frac{1}{D})}{2} \hat{H}(1, 1) + O(X^{-(M-1)}).$$

Then we note

$$\hat{H}(1, 1) = \int_0^\infty \int_0^\infty \int_0^\infty g(t) V_1\left(\frac{4\pi\sqrt{lt}}{x}\right) V_2\left(\frac{4\pi\sqrt{l't}}{y}\right) dt \frac{dx dy}{xy}.$$

A change of variables $x \rightarrow \frac{\sqrt{lt}}{4\pi x}, y \rightarrow \frac{\sqrt{l't}}{4\pi y}$, and the use of $\int_0^\infty g(x) dx = 1$, gives the result. \square

7.2. Evaluating $A_{n,X}, n \neq 0$. Fix r , then by Proposition 5.5, $X_n(r)$ is the set of c such that, setting $d = (c, c')$, we have

- (1) $\frac{c}{d}, \frac{c'}{d}$ are both prime to $\frac{n}{d\delta}$.
- (2) $\frac{\delta cr + \delta' c'}{d} \equiv 0 \pmod{\frac{n}{d}}$
- (3) $\frac{\delta cr + \delta' c'}{d} \not\equiv 0 \pmod{\frac{n}{k}}$ if k is a proper divisor of d .

Now for each divisor d of n , let $X_n(r, d)$ be the set of pairs c in $X_n(r)$ such that $(c, c') = d$.

We would like to prove that there is a constant $R(n, d)$ such that

$$(7.7) \quad \sum_{\mathfrak{m}} \sum_{c \in X_n(r, d)} \frac{1}{\mathbb{N}(c)} e\left(\frac{-1}{n} \left(\frac{lc'}{c} + \frac{l'c}{c'}\right)\right) I_{\mathfrak{m}}(n, c, X) = \left(1 + \frac{1}{D}\right) R(n, d) \frac{1}{n} \int_0^\infty \int_0^\infty H_n(x, y) dx dy + O(X^{-M}),$$

and

$$\sum_{d|n} R(n, d) = 1.$$

Remark. We want this expression to be independent of r as in the geometric side of the trace formula, the archimedean part does not depend on the r -sum inside of the Kloosterman sum.

Let us describe $X_n(r, d)$ explicitly. If $c \in X_n(r, d)$, then there exists a $\lambda \in \mathcal{O}_{\mathbf{K}}$ such that

$$(7.8) \quad \frac{\delta'c'}{d} = -\frac{\delta c}{d} r + \lambda \frac{n}{d}.$$

Let $a, b \in \mathcal{O}_{\mathbf{K}}$, then define $(a, b)_{\mathbb{Z}}$ to be the gcd of a and b inside of \mathbb{Z} .

Lemma 7.2. *Fix c such that $\frac{c}{d}$ is relatively prime to $\frac{c'}{d}$, and r such that $rr' \equiv 1(n)$. Let $\lambda \in \mathcal{O}_{\mathbf{K}}$ and defined by (7.8). Then $c \in X_n(r, d)$ if and only if $(\lambda, d)_{\mathbb{Z}} = 1$.*

Proof. If $(\frac{c}{d}, \frac{n}{d}) = p \neq 1$, then by (7.8), $p | \frac{c'}{d}$ a contradiction. And if p divides c'/d and n/d , then (7.8) gives $p | r(c/d)$. But $(r, n) = 1$, so this implies that p divides c/d – again a contradiction.

If $d = 1$, the requirements are met.

If $d \neq 1$, we must also require that

$$(7.9) \quad \frac{\delta cr + \delta'c'}{n} \not\equiv 0 \pmod{p \frac{n}{d}}$$

for all $p|d$. But

$$\frac{\delta cr + \delta'c'}{d} = \lambda \frac{n}{d}$$

Therefore (7.9) holds if and only if $\lambda \not\equiv 0 \pmod{p}$ for all $p|d$. In other words, λ must be relatively prime to d in the ring of integers of \mathbf{K} . But it suffices to check the gcd of λ and d in the rational integers. Now (7.8) implies there exists $q \in \mathcal{O}_{\mathbf{K}}$, such that

$$(7.10) \quad \lambda' = r'\lambda + \delta cq.$$

Suppose $(\lambda, d) = p \neq p'$ and $p' \nmid \lambda$, then (7.10) is contradicted. □

Perhaps it is more convenient to replace the c with dc where $1 = (c, c')$. Then the left-hand side of (7.7) is equal to

$$(7.11) \quad \sum_{\mathfrak{m}} \sum_{\substack{c: (c, c')=1 \\ cr - c' = \lambda \frac{n}{d\delta} \\ (\lambda, d)_{\mathbb{Z}}=1}} \frac{1}{\mathbb{N}(dc)} e\left(\frac{-1}{n} \left(\frac{lc'}{c} + \frac{l'c}{c'}\right)\right) I_{\mathfrak{m}}(n, dc, X).$$

To study this more we also need a further lemma counting solutions of (7.8).

Lemma 7.3. *Let $n \in \mathbb{Z}$ and $rr' \equiv 1(n)$. If $f|d$, $f \in \mathbb{Z}$, $d = d'$, and $d|n$, then the set*

$$\#\{c : cr - c' \equiv 0(\frac{nf}{d\delta}) \text{ and } (c, c') = 1\} = \frac{nf}{d}.$$

If $d = -d'$, then the set is of size $\frac{nf}{Dd}$.

Proof. We assume $d = d'$, the other case is similar. Let $c := a + b\sqrt{D}$ and $r := x + y\sqrt{D}$. Then the condition

$$(7.12) \quad cr \equiv c'(\frac{nf}{d\delta}),$$

implies $a(x-1) + Dby \equiv 0(\frac{nf}{d})$. As $D|n$, $(a, n) = 1$, implies $a = 0$ or $x \equiv 1(D)$. If $a = 0$, then $(c, c') > 1$, a contradiction.

Now assume $x \equiv 1(D)$, or $x = 1 + Dl$, $l \in \mathbb{N}$. Then $r = 1 + Dl + y\sqrt{D}$ and $rr' \equiv 1(n)$ implies

$$(7.13) \quad y^2 \equiv Dl^2 + 2l(h),$$

where $n = Dh$.

Likewise, (7.12) implies $al + by \equiv 0(\frac{hf}{d})$. Assume $a + b\sqrt{D}$ and $a + b'\sqrt{D}$ satisfy (7.12) for $b \neq b'$, then one gets $(b - b')l \equiv 0(\frac{Dhf}{d})$. If $l \equiv 0(\frac{Dhf}{d})$, then (7.13) implies $y \equiv 0(\frac{Dhf}{d})$, or $r \equiv 1(\frac{Dhf}{d})$. Thus counting the integral elements $\{c : c \equiv c'(\frac{nf}{d\delta})\}$ is clearly $\frac{nf}{d}$. Otherwise, $b \equiv b'(\frac{Dhf}{d})$, and again the size of the set is $\frac{nf}{d}$. The same argument works if we assume two pairs $a + b\sqrt{D}$ and $a' + b'\sqrt{D}$. □

Definition 7.4. *Let $H_{n,\mathfrak{m}}(x, x') := \frac{1}{xx'}e(\frac{xl'}{-nx'} + \frac{x'l}{-nx})I_{\mathfrak{m}}(n, x, 1)$.*

Like the $n = 0$ case, $H_{n,\mathfrak{m}}(x, x') = H_{n,1}(\mathfrak{m}x, \mathfrak{m}x')$, by a change of variables. Again for economy we label $H_n(x, x') = H_{n,1}(x, x')$. Then (7.11) equals

$$\frac{1}{X} \sum_{\mathfrak{m}} \sum_{\substack{c:(c,c')=1 \\ cr-c'=\lambda\frac{n}{d\delta} \\ (\lambda,d)_{\mathbb{Z}}=1}} H_n(\mathfrak{m}\frac{dc}{\sqrt{X}}, \mathfrak{m}\frac{dc'}{\sqrt{X}}).$$

Proposition 7.5. *For any integer $M \geq 0$,*

(1) *If $d = d'$,*

$$\frac{1}{X} \sum_{\mathfrak{m}} \sum_{\substack{c:(c,c')=1 \\ cr-c'=\lambda\frac{n}{d\delta} \\ (\lambda,d)_{\mathbb{Z}}=1}} H_n(\mathfrak{m}\frac{dc}{\sqrt{X}}, \mathfrak{m}\frac{dc'}{\sqrt{X}}) = \frac{1}{\sqrt{D}n} R(n, d) \int_0^\infty \int_0^\infty H_n(x, y) dx dy + O(n^{-3M} X^{-M}).$$

(2) *If $d = -d'$,*

$$\frac{1}{X} \sum_{\mathfrak{m}} \sum_{\substack{c:(c,c')=1 \\ cr-c'=\lambda\frac{n}{d\delta} \\ (\lambda,d)_{\mathbb{Z}}=1}} H_n(\mathfrak{m}\frac{dc}{\sqrt{X}}, \mathfrak{m}\frac{dc'}{\sqrt{X}}) = \frac{1}{D^{3/2}n} R(n, d) \int_0^\infty \int_0^\infty H_n(x, y) dx dy + O(n^{-3M} X^{-M}).$$

(3) $\sum_{d|n} R(n, d) = 1$

Corollary 7.6. $A_{n,X} = \frac{(1+\frac{1}{D})}{\sqrt{D}} \sum_{d|n} R(n,d) \int_0^\infty \int_0^\infty H_n(x,y) dx dy + O(n^{-3M} X^{-M}) = \frac{(1+\frac{1}{D})}{\sqrt{D}} \int_0^\infty \int_0^\infty H_n(x,y) dx dy + O(n^{-3M+\epsilon} X^{-M}),$ for $\epsilon > 0$.

Proof. {Corollary} Immediate after Proposition 7.5. \square

Proof. Take the case $d = d'$. We ignore the $\frac{1}{X}$ factor, and fix an \mathfrak{m} for the moment. The condition $(\lambda, d)_{\mathbb{Z}} = 1$ is now removed with Mobius inversion. Letting $E := cr - c'$, this gives

$$\sum_{(c,c')=1}^c \sum_{\substack{a|d \\ a|\frac{E}{\frac{n}{d\delta}}}} \mu(a) H_n(\mathfrak{m} \frac{dc}{\sqrt{X}}, \mathfrak{m} \frac{dc'}{\sqrt{X}}).$$

Rewritten this is

$$(7.14) \quad \sum_{a|d} \mu(a) \sum_{\substack{(c,c')=1 \\ E \equiv 0(\frac{na}{d\delta})}} H_n(\mathfrak{m} \frac{dc}{\sqrt{X}}, \mathfrak{m} \frac{dc'}{\sqrt{X}}).$$

The condition $E \equiv 0(\frac{na}{d\delta})$ is removed via Dirichlet characters. Note from Lemma 7.3, the size of the group of multiplicative characters on this group is $\phi(\frac{na}{d})$.

We get (7.14) equals

$$\sum_{\mathfrak{m}} \sum_{a|d} \mu(a) \sum_{(c,c')=1}^c \left(\frac{1}{\phi(\frac{na}{d})} \sum_{\chi(\frac{na}{d})} \chi(cr) \overline{\chi(c')} \right) H_n(\mathfrak{m} \frac{dc}{\sqrt{X}}, \mathfrak{m} \frac{dc'}{\sqrt{X}}).$$

Lastly, $(c, c') = 1$ is equivalent to $(c, l) = 1$ for all $l \in \mathbb{Z}$. This condition can be removed again by Mobius inversion. This provides

$$(7.15) \quad \sum_{\mathfrak{m}} \sum_{a|d} \mu(a) \sum_{\substack{l=1 \\ (l, \frac{na}{d})=1}}^{\infty} \mu(l) \sum_c \left(\frac{1}{\phi(\frac{na}{d})} \sum_{\chi(\frac{na}{d})} \chi(lcr) \overline{\chi(lc')} \right) H_n(\mathfrak{m} \frac{dlc}{\sqrt{X}}, \mathfrak{m} \frac{dlc'}{\sqrt{X}}).$$

The main contribution comes from the principal character $\chi_0(\frac{na}{d})$. We label the term with the principal character as $T^M(n, d)$ and the rest of the characters as $T^R(n, d)$. Therefore, (7.15) equals $T^M(n, d) + T^R(n, d)$.

7.3. Analysis of $T^M(n, d)$. We focus now on $T^M(n, d)$. This term fully written out is

$$(7.16) \quad \sum_{\mathfrak{m}} \sum_{a|d} \frac{\mu(a)}{\phi(\frac{na}{d})} \sum_{\substack{l=1 \\ (l, \frac{na}{d})=1}}^{\infty} \mu(l) \sum_{(c, \frac{na}{d})=1} H_n(\mathfrak{m} \frac{dlc}{\sqrt{X}}, \mathfrak{m} \frac{dlc'}{\sqrt{X}}).$$

Now we aim to perform Poisson summation on the c -sum. First we notice in (7.14) one cannot have an inert prime p divide c , or analogously the ramified prime \sqrt{D} cannot divide c . Thus the condition $(c, \frac{na}{d}) = 1$ can be restricted to only split primes, which we label $(c, \frac{na}{d})_{Spl} = 1$. With Poisson summation we have,

$$(7.17) \quad \sum_{(c, \frac{na}{d})_{Spl}=1} H_n(\mathfrak{m} \frac{dlc}{\sqrt{X}}, \mathfrak{m} \frac{dlc'}{\sqrt{X}}) = \frac{1}{\sqrt{D}} \prod_{p|\frac{na}{d}} (1-1/p)^2 \sum_{m \in \mathcal{O}_{\mathbf{K}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(\mathfrak{m} \frac{dlx}{\sqrt{X}}, \mathfrak{m} \frac{dly}{\sqrt{X}}) e(mx + m'y) dx dy.$$

The factor $\frac{1}{\sqrt{D}}$ comes from the volume factor of the summation formula over number fields, see [V] for a statement of the formula.

Including this into (7.16) and a change of variables $x \rightarrow \frac{\sqrt{X}x}{\mathfrak{m}dl}$ and $y \rightarrow \frac{\sqrt{X}y}{\mathfrak{m}dl}$, we get

$$(7.18) \quad \frac{X}{\sqrt{D}d^2} \sum_{\mathfrak{m}} \frac{1}{\mathfrak{m}^2} \sum_{a|d} \frac{\mu(a)}{\phi(\frac{na}{d})} \sum_{\substack{l=1 \\ (l, \frac{na}{d})=1}}^{\infty} \frac{\mu(l)}{l^2} \prod_{p|\frac{na}{d}} (1-1/p)^2 \sum_{m \in \mathcal{O}_{\mathbf{K}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) e\left(\frac{X}{ld}(mx+m'y)\right) dx dy.$$

Now $H_n(x, y)$ is compactly supported away from zero, and integration by parts Q -times for $m \neq 0$, one gets

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) e\left(\frac{X}{ld}(mx+m'y)\right) dx dy = \left(\frac{ld}{mX}\right)^Q \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^Q H_n(x, y)}{\partial x^Q} e\left(\frac{X}{ld}(mx+m'y)\right) dx dy.$$

To simplify (7.18), we define

$$(7.19) \quad B_m(n, d, X) := \frac{1}{\sqrt{D}X^Q} \sum_{\mathfrak{m}} \frac{1}{\mathfrak{m}^{2-Q}} \frac{1}{d^{2-Q}m^Q} \sum_{a|d} \frac{\mu(a)}{\phi(\frac{na}{d})} \sum_{\substack{l=1 \\ (l, \frac{na}{d})=1}}^{\infty} \frac{\mu(l)}{l^{2-Q}} \prod_{p|\frac{na}{d}} (1-1/p)^2 \times \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^Q H_n(x, y)}{\partial x^Q} e\left(\frac{X}{ld}(mx+m'y)\right) dx dy.$$

Now remember that in Definition 7.4 of $H_n(x, x')$ is

$$I_{\mathfrak{m}}(n, x, X) = \int_{-\infty}^{\infty} e\left(\frac{Xtn}{\mathbf{N}(\delta x)}\right) g(\mathfrak{m}^2 t) V_1\left(\frac{4\pi\sqrt{X}tl}{x}\right) V_2\left(\frac{4\pi\sqrt{X}tl'}{x'}\right) dt.$$

Again, integration by parts K -times in this integral gives the bound $O(\frac{1}{n^K})$, and thus we also have the same bound for $H_n(x, x')$.

We recall that

$$\sum_{\substack{l=1 \\ (l, \frac{na}{d})=1}}^{\infty} \frac{\mu(l)}{l^{2s}} = \frac{1}{\zeta(2s)} \prod_{p|\frac{na}{d}} \frac{1}{(1 - \frac{1}{p^{2s}})}.$$

Notice now that then the product of the \mathfrak{m} and l sum is $\prod_{p|\frac{na}{d}} \frac{1}{(1 - \frac{1}{p^{2-Q}})}$.

Using trivial bounds on the $a|d$ -sum and $p|\frac{na}{d}$ -sum, and taking $K \geq 3Q$, we get

$$(7.20) \quad \sum_{m \neq 0 \in \mathcal{O}_{\mathbf{K}}} B_m(n, d, X) = O(n^{-3Q} X^{-Q}).$$

Now we focus on the $m = 0$ term. Again the \mathfrak{m} -sum and the l -sum cancel, so the term to compute is

$$\frac{X}{\sqrt{D}d^2} \sum_{a|d} \frac{\mu(a)}{\phi(\frac{na}{d})} \prod_{p|\frac{na}{d}} \left(\frac{1 - \frac{1}{p}}{1 + \frac{1}{p}}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) dx dy.$$

Using $\frac{1}{\phi(n)} \prod_{p|n} (1 - \frac{1}{p}) = \frac{1}{n}$, our expression becomes $\frac{XR(n,d)}{\sqrt{D}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) dx dy$, where

$$(7.21) \quad R(n, d) := \frac{1}{dn} \sum_{a|d} \frac{\mu(a)}{a} \prod_{p|\frac{na}{d}} \frac{1}{(1 + \frac{1}{p})}.$$

Lemma 7.7. $\sum_{d|n} R(n, d) = \frac{1}{n}$.

Proof. $R(n, d)$ is multiplicative and so it remains to evaluate it at a prime power $n = p^l$ where $d = p^i, i = 0, \dots, l$. It is sufficient to do this computation for $i = 0, 1 \leq i \leq l-1$, and $i = l$. For $i = 0$, the sum has only one term

$$R(p^l, 1) = \frac{1}{p^l(1 + \frac{1}{p})}.$$

For $1 \leq i \leq l-1$,

$$R(p^l, p^i) = \frac{1}{p^{l+i}} \left(\frac{1 - \frac{1}{p}}{1 + \frac{1}{p}} \right).$$

For $i = l$,

$$R(p^l, p^l) = \frac{1}{p^{2l}} \left(1 - \frac{1}{p(1 + \frac{1}{p})} \right) = \frac{1}{p^{2l}(1 + \frac{1}{p})}.$$

Now

$$\sum_{i=1}^{l-1} \frac{1}{p^i} = \frac{1 - \frac{1}{p^l}}{1 - \frac{1}{p}} - 1 = \frac{\frac{1}{p} - \frac{1}{p^l}}{1 - \frac{1}{p}}.$$

So

$$(7.22) \quad \sum_{i=0}^l R(p^l, p^i) = \left[\frac{1}{p^l(1 + \frac{1}{p})} \right] + \left[\frac{1}{p^l} \left(\frac{1 - \frac{1}{p}}{1 + \frac{1}{p}} \right) \frac{\frac{1}{p} - \frac{1}{p^l}}{1 - \frac{1}{p}} \right] + \left[\frac{1}{p^{2l}(1 + \frac{1}{p})} \right]$$

$$(7.23) \quad = \frac{1}{p^{2l}(1 + \frac{1}{p})} [p^l + p^{l-1} - 1 + 1]$$

$$(7.24) \quad = \frac{p^l + p^{l-1}}{p^{2l}(1 + \frac{1}{p})} = \frac{1}{p^l}.$$

□

Incorporating back the $\frac{1}{X}$ factor we are left with

$$\frac{1}{\sqrt{D}n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) dx dy.$$

But remember $H_n(x, y)$ is defined in terms of the test functions $V_1, V_2 \in C_0^\infty(\mathbb{R}^+)$, so integration is limited to $\int_0^\infty \int_0^\infty H_n(x, y) dx dy$. The same computation can be done for $d = -d'$, to get

$$\frac{1}{D^{3/2}n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) dx dy.$$

7.4. **Analysis of $T^R(n, d)$.** Here

(7.25)

$$T^R(n, d) := \sum_{\mathfrak{m}} \sum_{a|d} \frac{\mu(a)}{\phi(\frac{na}{d})} \sum_{\chi \neq \chi_0(\frac{na}{d})} \sum_{\substack{l=1 \\ (l, \frac{na}{d})=1}}^{\infty} \mu(l) \sum_c \left(\frac{1}{\phi(\frac{na}{d})} \chi(lcr) \overline{\chi(lc')} \right) H_n(\mathfrak{m} \frac{dlc}{\sqrt{X}}, \mathfrak{m} \frac{dlc'}{\sqrt{X}}).$$

Now fix a character $\chi \neq \chi_0$ and break the c -sum into arithmetic progressions modulo $(\frac{na}{d})$ to get the inner sum equals

$$(7.26) \quad \sum_{q(\frac{na}{d})} \chi(q) \overline{\chi(q')} \sum_{c \equiv q(\frac{na}{d})} H_n(\mathfrak{m} \frac{dlc}{\sqrt{X}}, \mathfrak{m} \frac{dlc'}{\sqrt{X}}).$$

Note we used $|\chi(l)|^2 = 1$. Now using Poisson summation on $c = q + \frac{na}{d}m, m \in \mathcal{O}_{\mathbf{K}}$, we get, after a few change of variables, (7.26) equaling

$$\frac{X}{(\mathfrak{m}nla)^2} \sum_{q(\frac{na}{d})} \chi(q) \overline{\chi(q')} \sum_{m \in \mathcal{O}_{\mathbf{K}}} e(T r_{\mathbf{K}/\mathbf{Q}}(\frac{mq}{d})) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_n(x, y) e(\frac{X}{ld}(mx + m'y)) dx dy.$$

Now similar to the poisson summation for $T^M(n, d)$, we separate the sum into cases $m = 0$ and $m \neq 0$. Again, the interchange of sums is valid due to $H_n(x, y)$ being smooth and compactly supported.

Now for $m = 0$, we execute the q -sum which is zero since it is a complete character sum. As for $m \neq 0$, the exact same analysis used to get (7.20) can be reused for $T^R(n, d)$. Therefore, for any $Q \geq 0$,

$$T^R(n, d) = O(n^{-3Q} X^{-Q}).$$

This completes Proposition 7.5. □

8. FROM EXPONENTIAL SUMS OVER A QUADRATIC FIELD TO KLOOSTERMAN SUMS

In order to classify base change, one needs a bridge between a trace formula over our quadratic field \mathbf{K} to a trace formula over \mathbb{Q} . This formula was discovered in [Z].

We need a few definitions first. Let $D = D_1 D_2$, then define $\psi(D_2) = (\frac{D_1}{D_2}) \sqrt{D_2}$. Next define

$$H_b(n, m) = \sum_{\substack{D=D_1 D_2 \\ D_2 | n \\ (b, D_2)=1}} \frac{\psi(D_2)}{D_2} H_{bD_1}^{D_1}(\frac{n}{D_2}, m),$$

where

$$H_c^{D_1}(n, m) = \frac{1}{c} (\frac{c}{D_2}) S_{D_1}(n \overline{D_2}, m, c),$$

and

$$S_{D_1}(n \overline{D_2}, m, c) = \sum_{d(c)^*} (\frac{d}{D_1}) e(\frac{n \overline{D_2} d + md}{c}).$$

Proposition 8.1. *Let D be prime, then writing $n = Da, a \in \mathbb{N}$*

$$\sum_{\substack{r \in \mathcal{O}_{\mathbf{K}}/(\frac{n}{D}) \\ rr' \equiv 1(n)}} e(\frac{rl + r'l'}{n}) = a\sqrt{D} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} H_{a/r}(-\frac{ll'}{r^2}, -1).$$

Corollary 8.2. *If $(l, D) = 1$ then Proposition 8.1 implies*

$$\sum_{\substack{r \in \mathcal{O}_{\mathbf{K}}/(\frac{n}{\delta}) \\ rr' \equiv 1(n)}} e\left(\frac{rl + r'l'}{n}\right) = \frac{1}{\sqrt{D}} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D\left(\frac{ll'}{r^2}, 1, \frac{Da}{r}\right).$$

9. THE CONTINUOUS SPECTRUM

We now need to compute the continuous spectrum contribution on the spectral side of the trace formula over the quadratic field \mathbf{K} . Unfortunately, the normalization for the Fourier coefficients for the Eisenstein series is formidable.

Let

$$\omega(x) := \left| \frac{x}{x'} \right|^{\frac{i\pi k}{2 \log \epsilon_0}}, k \in \mathbb{Z},$$

where ϵ_0 is the fundamental unit of our field \mathbf{K} . From [BMP2], we let

$$(9.1) \quad \Omega(r, 1/2 + it, i\mu) = \left[\frac{2\mathbb{N}(r)^{it}}{\sqrt{\cosh(\pi(t + \mu)) \cosh(\pi(t + \mu'))}} \right] \frac{|\pi|^{i(t+\mu)}}{\Gamma(1/2 + i(t + \mu))\Gamma(1/2 + i(t + \mu'))} \times \\ \sum_{(c)} \frac{\omega^2(c) S(r, 0, c)}{\mathbb{N}(c)^{1+2it}},$$

where $S(r, 0, c) = \sum_{x(c)^*} e(xr/c)$.

The continuous spectrum contribution is

$$(9.2) \quad CSC_{\rho, \xi} := \frac{(1 + \frac{1}{D})}{L(1, \chi_D)} \sum_{\mu} \int_{-\infty}^{\infty} h(V_1, t + \mu) h(V_2, t + \mu') \Omega(\rho, 1/2 + it, i\mu) \overline{\Omega(\xi, 1/2 + it, i\mu')} dt.$$

More specifically, μ is in the lattice $Tr_{\mathbf{K}/\mathbb{Q}}(x) = 0$, such that $|\epsilon|^{i\mu} = 1$, for all $\epsilon \in \mathcal{O}^*$. In our case, $\mu = \frac{\pi k}{2 \log \epsilon_0}$, $k \in \mathbb{Z}$.

We prove in this section

Proposition 9.1. *If*

$$\sigma_{l, \omega}(n) := \sum_{a|n} \omega(a) \mathbb{N}(a)^l,$$

$$(9.3) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) CSC_{n, \xi} = (1 + \frac{1}{D}) \left(\sum_{\mu} \frac{h(V_1, \mu) h(V_2, \mu)}{\pi^2 L(1, \chi_D) L(1, \omega_{\mu}^2)} \omega(\xi)^{-2} \sigma_{0, \omega_{\mu}^2}(\xi) + \right.$$

$$(9.4) \quad \left. \int_{-\infty}^{\infty} h(V_1, t) h(V_2, t) \frac{\sigma_{-2it, 0}(\xi) \mathbb{N}(\xi)^{it}}{\cosh(\pi t)^2 |\Gamma(1/2 + it)|^4 |L(1 - 2it, \chi_D)|^2} dt \right) + O(X^{-M}),$$

for any integer $M \geq 0$.

We aim first to simplify our normalized coefficients Ω . Using Mobius inversion

$$(9.5) \quad \sum_{(c)} \frac{\omega^2(c) S(r, 0, c)}{\mathbb{N}(c)^{1+2it}} = \frac{\omega(r)^{-2} \sigma_{2it, \omega^2}(r)}{\mathbb{N}(r)^{2it} L(1 + 2it, \omega^2)}.$$

Noting for $n \in \mathbb{N}$, $\omega(n)^2 = 1$, we collect the n and \mathfrak{m} terms together from (9.2) to get

$$(9.6) \quad \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) \frac{\sigma_{2it, \omega^2}(n)}{\mathbb{N}(n)^{it}} = \frac{1}{2\pi i} \int_{(\sigma)} G(s) X^s \zeta(2s) L(s) ds,$$

for $\sigma > 0$ sufficiently large. Here

$$L(s) := \sum_{n=1}^{\infty} \frac{\sigma_{2it, \omega^2}(n)}{n^{s+2it}},$$

is the Asai L-function associated to the Eisenstein series over the quadratic field \mathbf{K} . Using the multiplicativity of the divisor function, (9.6) equals for p split, $p = \mathbf{p}_1 \mathbf{p}_2$,

$$\frac{(1 - \frac{1}{p^{2s}})}{(1 - \frac{\omega^2(\mathbf{p}_1)}{p^s})(1 - \frac{\omega^2(\mathbf{p}_2)}{p^s})(1 - \frac{1}{p^{s+2it}})(1 - \frac{1}{p^{s-2it}})}.$$

For p inert, $\omega^2(p) = 1$, so we get

$$\frac{1}{(1 - \frac{1}{p^{s+2it}})(1 - \frac{1}{p^{s-2it}})} = \frac{(1 - \frac{1}{p^{2s}})}{(1 - \frac{1}{p^s})(1 - \frac{\chi_D(p)}{p^s})(1 - \frac{1}{p^{s+2it}})(1 - \frac{1}{p^{s-2it}})}.$$

For $p = \mathbf{p}_1^2$ ramified, $\chi(\mathbf{p}_1) = 1$, and likewise,

$$\frac{(1 + \frac{1}{p^s})}{(1 - \frac{1}{p^{s+2it}})(1 - \frac{1}{p^{s-2it}})} = \frac{(1 - \frac{1}{p^{2s}})}{(1 - \frac{1}{p^s})(1 - \frac{\chi_D(p)}{p^s})(1 - \frac{1}{p^{s+2it}})(1 - \frac{1}{p^{s-2it}})}.$$

We now have 2 cases: $k \neq 0$ and $k = 0$.

9.1. $\mathbf{k} \neq \mathbf{0}$. Its clear $L(s)$ has poles at $s = 1 \pm 2it$, but as well the $\zeta(2s)$ cancels with the $\zeta(2s)$, in the Asai L-function.

Thus, shifting the contour of the integral to $-M$, $M \geq 0$ we pick up a pole at $s = 1 + 2it$ with residue

$$X^{1+2it} \frac{\zeta(1+4it) L(1+2it, \omega^2)}{\zeta(2(1+2it))}.$$

The analogous argument follows for the pole at $s = 1 - 2it$.

Now using the contour shift (using the fact we chose $G(1) = 1$.) and the functional equation of the gamma function

$$(9.7) \quad \Gamma(1/2 + it) \Gamma(1/2 - it) = \frac{\pi}{\cosh \pi t},$$

we get a term with $k \neq 0$ on the left hand side of Proposition 9.1 is

$$(9.8) \quad \frac{1}{X} \sum_{n, \mathfrak{m}} g(\mathfrak{m}^2 n / X) \frac{(1 + \frac{1}{D})}{L(1, \chi_D)} \int_{-\infty}^{\infty} h(V_1, t + \mu) h(V_2, t + \mu') \Omega(n, 1/2 + it, i\mu) \overline{\Omega(\xi, 1/2 + it, i\mu)} dt =$$

$$\frac{(1 + \frac{1}{D})}{L(1, \chi_D)} \int_{-\infty}^{\infty} h(V_1, t + \mu) h(V_2, t + \mu') \times$$

$$\left(\frac{\omega(\xi)^{-2} \sigma_{-2it, \omega^2}(\xi) \mathbb{N}(\xi)^{it}}{\cosh(\pi(t + \mu)) \cosh(\pi(t + \mu')) |\Gamma(1/2 + i(t + \mu)) \Gamma(1/2 + i(t + \mu'))|^2 |L(1 + 2it, \omega^2)|^2} \right) \times$$

$$X^{2it} \zeta(1 + 4it) L(1 + 2it, \omega^2) dt + O(X^{-M}).$$

We note the term $\frac{1}{L(1, \chi_D)}$, is mentioned in [J], but buried as a constant in [BMP1].

Here $\zeta(1 + 4it)$ has a pole at $t = 0$, and to understand this we use the following lemma.

Lemma 9.2. *Let H be a differentiable function in $L^1(\mathbb{R})$, then*

$$PV \int_{-\infty}^{\infty} H(x) e^{ikx} \frac{dx}{x} := \lim_{\rightarrow 0^+} \int_{|x| \geq} H(x) e^{ikx} \frac{dx}{x} \rightarrow \pm \pi i H(0) \text{ as } \pm k \rightarrow \infty.$$

Proof. See [V], Lemma 10. □

Applying this lemma for $k = -\log X$, we obtain (9.8) equals

$$(9.9) \quad \frac{(1 + \frac{1}{D})h(V_1, \mu)h(V_2, \mu)}{L(1, \chi_D) \cosh(\mu) \cosh(\mu')} \left(\frac{\omega(\xi)^{-2} \sigma_{0, \omega^2}(\xi)}{|\Gamma(1/2 + i\mu)\Gamma(1/2 + i\mu')|^2 L(1, \omega_\mu^2)} \right) + O(X^{-M}).$$

Here we also used $\mu' = -\mu$, and that $h(V_j, t)$ is real valued function. Using (9.7) again we get for $k \neq 0$

$$\frac{(1 + \frac{1}{D})}{\pi^2 L(1, \chi_D) L(1, \omega^2)} h(V_1, \mu) h(V_2, \mu) \omega(\xi)^{-2} \sigma_{0, \omega_\mu^2}(\xi) + O(X^{-M}).$$

9.2. **k = 0.** Loosely, the same calculations go as the previous section but here we have

$$L(s) = \frac{\zeta(s + 2it) \zeta(s - 2it) \zeta(s) L(s, \chi_D)}{\zeta(2s)}.$$

Again the $\zeta(2s)$ cancels with the m -sum, and the poles of $L(s)$ are at $s = 1, 1 \pm 2it$. However, the poles at $s = 1 \pm 2it$, are cancelled by

$$\frac{1}{|L(1 + 2it, \omega_0^2)|^2} = \frac{1}{|\zeta(1 + 2it) L(1 + 2it, \chi_D)|^2}$$

found in (9.5) after expanding $\Omega(r, 1/2 + it, 0) \Omega(\xi, 1/2 + it, 0)$. Now for $k = 0$, the left hand side of Proposition 9.1 after a contour shift to $-M, M \geq 0$, similar to (9.6),

$$(9.10) \quad \frac{(1 + \frac{1}{D})}{XL(1, \chi_D)} \int_{-\infty}^{\infty} dt h(V_1, t) h(V_2, t) \left(\frac{\sigma_{-2it, 0}(\xi) \mathbb{N}(\xi)^{it}}{\cosh(\pi t)^2 |\Gamma(1/2 + it)|^4 |\zeta(1 + 2it) L(1 + 2it, \chi_D)|^2} \right) \times \\ \left[G(1) X |\zeta(1 + 2it)|^2 L(1, \chi_D) + G(1 + 2it) X^{1+2it} \zeta(1 + 4it) \zeta(1 + 2it) L(1 + 2it, \chi_D) \right. \\ \left. + G(1 - 2it) X^{1-2it} \zeta(1 - 4it) \zeta(1 - 2it) L(1 - 2it, \chi_D) \right]$$

We deal with the terms containing $X^{1 \pm 2it}$ first. Take the term with X^{1+2it} , the other term will be analogous. This term is after some simplifications,

$$(9.11) \quad \frac{(1 + \frac{1}{D})}{L(1, \chi_D)} \int_{-\infty}^{\infty} h(V_1, t) h(V_2, t) \left(\frac{\sigma_{-2it, 0}(\xi) \mathbb{N}(\xi)^{it} G(1 + 2it) X^{2it} \zeta(1 + 4it)}{\cosh(\pi t)^2 |\Gamma(1/2 + it)|^4 \zeta(1 - 2it) L(1 - 2it, \chi_D)} dt \right)$$

Now $\frac{\zeta(1+4it)}{\zeta(1-2it)}$ is analytic for $t \in \mathbb{R}$, so

$$F(t) := h(V_1, t) h(V_2, t) \left(\frac{\sigma_{-2it, \chi}(0) \mathbb{N}(\xi)^{it} G(1 + 2it) \zeta(1 + 4it)}{\cosh(\pi t)^2 |\Gamma(1/2 + it)|^4 \zeta(1 - 2it) L(1 - 2it, \chi_D) \zeta(2(1 + 2it))} \right)$$

is analytic and

$$\int_{-\infty}^{\infty} F(t)X^{2it}dt \ll O(X^{-M}),$$

for any $M > 0$.

Now, we address the term in (9.10) coming from the pole at $s = 1$. Similar to (9.11), we simplify to get

$$(9.12) \quad (1 + \frac{1}{D}) \int_{-\infty}^{\infty} h(V_1, t)h(V_2, t) \left(\frac{\sigma_{-2it, 0}(\xi)\mathbb{N}(\xi)^{it}}{\cosh(\pi t)^2 |\Gamma(1/2 + it)|^4 |L(1 - 2it, \chi_D)|^2} dt \right) + O(X^{-M}).$$

10. PUTTING IT ALL TOGETHER

Incorporating Proposition 7.1, Corollary 7.6, and Corollary 8.2, we have (6.10),

$$(10.1) \quad (L) = \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) \left(\sum_{\Pi \neq 1} h(V, \nu_{\Pi}) c_n(\Pi) \overline{c_l(\Pi)} + C S C_{n, l} \right) =$$

$$\frac{(1 + \frac{1}{D})}{2} [\delta(l, l') \int_0^{\infty} \int_0^{\infty} V_1(x) V_2(y) \frac{dx dy}{xy} +$$

$$+ \frac{1}{D} \sum_{a=1}^{\infty} \frac{1}{Da} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D(\frac{ll'}{r^2}, 1, \frac{Da}{r}) \int_0^{\infty} \int_0^{\infty} H_{Da}(x, y) dx dy] + O(X^{-M}).$$

We only mention here that we used Proposition 7.5 to get

$$\sum_{a=1}^{\infty} O((Da)^{-3M} X^{-M}) = O(X^{-M}),$$

for M large.

Our aim now is to show (L) is a sum of the geometric sides of the Kuznetsov and Petersson's trace formula. Each one then gives a spectral side from which we then gather information about poles of the Asai L-function and hence information about base change.

We now follow section 9 of [H] closely. Define

$$(10.2) \quad V_1 * V_2(z) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(z \frac{i}{2} \left(\frac{x}{y} + \frac{y}{x}\right)\right) \exp\left(\left(\frac{1}{z}\right) \frac{8\pi^2 i}{xy}\right) \times$$

$$V\left(\frac{4\pi}{x}\right) W\left(\frac{4\pi}{y}\right) \frac{dx dy}{x y}.$$

Recall Definition 6.3 and the dependence on l , we get

$$(10.3) \quad \int_0^{\infty} \int_0^{\infty} H_{Da}(x, y) dx dy = D \left(V_1 * V_2\left(\frac{4\pi\sqrt{ll'}}{Da}\right) \right).$$

The D factor comes from a change of variables $t \rightarrow Dt$ in the definition of $I_m(n, x, 1)$ inside of Definition 6.3.

Now Theorem 3.1 of [H] states

Theorem 10.1. *For all $V, W \in C_0^{\infty}(\mathbb{R}^+)$, $h(V * W, t) = C_t h(V, t) h(W, t)$, where $C_t = 2\pi$ for t an even integer, and $C_t = \pi$ for t purely imaginary.*

Let

$$M(t) = h(V_1, t)h(V_2, t),$$

then using Sears-Titchmarsh inversion formula for $V_1 * V_2(z)$, see [H], we get

$$(10.4) \quad V_1 * V_2(z) = 4\pi \left(\int_0^\infty M(t) \tanh(\pi t) B_{2it}(z) t dt + \sum_{2k>0, k \in \mathbb{N}} (k-1) J_{k-1}(z) M(k) \right),$$

Further from [H] we need

Proposition 10.2. *Let $M(t) = h(V_1, t)h(V_2, t)$ then,*

$$(10.5) \quad \int_0^\infty V_1(x) V_2(x) \frac{dx}{x} = 2 \left(\int_{-\infty}^\infty M(t) \tanh(\pi t) t dt + \sum_{2k>0, k \in \mathbb{N}} (k-1) M(k) \right).$$

The limit (L) can be then rewritten as

$$(10.6) \quad (L) = 2\pi(1 + \frac{1}{D}) \left[\delta(l, l') \left(\int_{-\infty}^\infty M(t) \tanh(\pi t) t dt + \sum_{2k>0, k \in \mathbb{N}} (k-1) M(k) \right) + \left(\sum_{a=1}^\infty \frac{1}{Da} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D\left(\frac{ll'}{r^2}, 1, \frac{Da}{r}\right) \left(\int_0^\infty M(t) \tanh(\pi t) B_{2it}\left(\frac{4\pi\sqrt{ll'}}{Da}\right) t dt + \sum_{2k>0, k \in \mathbb{N}} (k-1) J_{k-1}\left(\frac{4\pi\sqrt{ll'}}{Da}\right) M(k) \right) \right] \right].$$

Theorem 10.3. *(Kuznetsov trace formula) Denote the Maass form of eigenvalue $1/4 + t^2$ and level D and nebentypus χ_D by $\phi_{t,D}$, and let $\eta(l, 1/2 + it) := 2\pi^{1+it} \cosh(\pi t)^{-1/2} \frac{\tau_{D, it}(n)}{\Gamma(1/2 + it)L(1 + 2it, \chi_D)}$, where $\tau_{it}(n) = \sum_{ab=n} \chi_D(a)(a/b)^{it}$. Let $S_D(n, m, c) = \sum_{x(c)^*} \chi_D(x) e(\frac{nx + m\bar{x}}{c})$. Then the Kuznetsov trace formula for these Maass forms and its associated continuous spectrum (See [Iw]) is*

$$(10.7) \quad \sum_{\phi_t} G(t_\phi) a_l(\phi_t) \overline{a'_l(\phi_t)} + \frac{1}{4\pi} \int_{-\infty}^\infty G(t) \eta(l, 1/2 + it) \overline{\eta(l', 1/2 + it)} dt = \\ = G_0 + \sum_{D|c} \frac{S_D(l, l', c)}{c} G^+(4\pi\sqrt{ll'}/c),$$

where $G_0 := \frac{\delta(l, l')}{\pi} \int_{-\infty}^\infty G(t) \tanh(\pi t) t dt$, and $G^+(x) := 4 \int_0^\infty G(t) \tanh(\pi t) B_{2it}(x) t dt$.

Theorem 10.4. *(Petersson trace formula) Let the holomorphic forms of weight k , level D , and nebentypus χ_D be denoted as $\phi_{k,D}$. Petersson's trace formula then states*

$$(10.8) \quad \sum_{\phi_k} G(k_\phi) a_l(\phi_k) \overline{a'_l(\phi_k)} = \frac{\delta(l, l')}{\pi} \sum_{2k>0, k \in \mathbb{N}} (k-1) G(k) + \sum_{D|c} \frac{S_D(l, l', c)}{c} \hat{G}(4\pi\sqrt{ll'}/c),$$

where

$$(10.9) \quad \hat{G}(x) = 4 \sum_{k>0, k \text{ even}} (k-1) G(k) J_{k-1}(x).$$

Notice that (10.6) almost looks like the geometric sides of these two trace formulas.

We first need to simplify the Kloosterman sums in (10.6). Inversion of the r - and s - sums give

$$\begin{aligned}
 (10.10) \quad & \sum_{a=1}^{\infty} \frac{1}{Da} \sum_{\substack{r \in \mathbb{N} \\ r|l \\ r|a}} r S_D\left(\frac{ll'}{r^2}, 1, \frac{Da}{r}\right) \int_0^{\infty} M(t) \tanh(\pi t) B_{2it}\left(\frac{4\pi\sqrt{ll'}}{Da}\right) t dt = \\
 & \sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{a=1}^{\infty} \frac{1}{Da} S_D\left(\frac{ll'}{r^2}, 1, Da\right) \int_0^{\infty} M(t) \tanh(\pi t) B_{2it}\left(\frac{4\pi\sqrt{ll'}}{Dra}\right) t dt = \\
 & \sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{a=1}^{\infty} \frac{1}{Da} S_D\left(\frac{ll'}{r^2}, 1, Da\right) \int_0^{\infty} M(t) \tanh(\pi t) B_{2it}\left(\frac{4\pi\sqrt{\frac{ll'}{r^2}}}{Da}\right) t dt
 \end{aligned}$$

We rewrite the holomorphic side of the trace formula as well in this fashion in (10.6).

Using the spectral sides of the Kuznetsov and Petersson trace formula we get (L) equals,

$$\begin{aligned}
 (10.11) \quad & 2\pi\left(1 + \frac{1}{D}\right) \sum_{\substack{r \in \mathbb{N} \\ r|l}} \left[\delta(l, l') \int_{-\infty}^{\infty} M(t) \tanh(\pi t) t dt + \sum_{a=1}^{\infty} \frac{1}{Da} S_D\left(\frac{ll'}{r^2}, 1, Da\right) \int_0^{\infty} M(t) \tanh(\pi t) B_{2it}\left(\frac{4\pi\sqrt{\frac{ll'}{r^2}}}{Da}\right) t dt \right. \\
 & \left. + \delta(l, l') \sum_{2k>0, k \in \mathbb{N}} (k-1) M(k) + \sum_{a=1}^{\infty} \frac{1}{Da} S_D\left(\frac{ll'}{r^2}, 1, Da\right) \sum_{2k>0, k \in \mathbb{N}} (k-1) J_{k-1}\left(\frac{4\pi\sqrt{\frac{ll'}{r^2}}}{Da}\right) M(k) \right] = \\
 & = 2\pi\left(1 + \frac{1}{D}\right) \sum_{\substack{r \in \mathbb{N} \\ r|l}} \left(\sum_{\phi_{t,D}} M(t_{\phi}) a_{\frac{ll'}{r^2}}(\phi_{t,D}) \overline{a_1(\phi_{t,D})} + \right. \\
 & \quad \left. \frac{1}{4\pi} \int_{-\infty}^{\infty} M(t) \eta\left(\frac{ll'}{r^2}, 1/2 + it\right) \overline{\eta(1, 1/2 + it)} dt + \sum_{\phi_{k,D}} M(k_{\phi}) a_{\frac{ll'}{r^2}}(\phi_k) \overline{a_1(\phi_k)} \right).
 \end{aligned}$$

We showed in Section 9,

$$\begin{aligned}
 (10.12) \quad & \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) C S C_{n,l} = 2\pi\left(1 + \frac{1}{D}\right) \left(\sum_{\mu} \frac{h(V_1, \mu) h(V_2, \mu)}{L(1, \chi_D) L(1, \omega_{\mu}^2)} \omega(l)^{-2} \sigma_{0, \omega_{\mu}^2}(l) + \right. \\
 & \quad \left. \int_{-\infty}^{\infty} h(V_1, t) h(V_2, t) \frac{\sigma_{-2it, 0}(l) \mathbb{N}(l)^{2it}}{\cosh(\pi t)^2 |\Gamma(1/2 + it)|^4 |L(1 - 2it, \chi_D)|^2} dt \right) + O(X^{-M}),
 \end{aligned}$$

for $M > 0$.

10.1. Comparing Coefficients from $\mathbb{Q}(\sqrt{D})$ to \mathbb{Q} . In this section we compare the coefficients of the continuous spectrum from (10.12) to those of (10.11). With the comparison we then show the continuous spectrum from (10.11) equals (10.12) minus the sum over the lattice μ .

Further we show the μ -sum in (10.12) is associated with a sub sum ("Theta functions") of the cuspidal spectrum in (10.11).

Define $\psi_\mu(k) := \sum_{\mathbb{N}(q)=k} \omega_\mu(q)$.

Lemma 10.5.

- (1) $\mathbb{N}(l)^{it} \sigma_{-2it,0}(l) = \sum_{\substack{r \in \mathbb{N} \\ r|l}} \tau_{it}\left(\frac{ll'}{r^2}\right),$
- (2) $\omega_\mu(l)^{-2} \sigma_{0,\omega_\mu^2}(l) = \sum_{\substack{r \in \mathbb{N} \\ r|l}} \psi_\mu\left(\frac{ll'}{r^2}\right).$

Proof. By multiplicativity, we do this for a prime power. Notice

$$\mathbb{N}(p^m)^{2it} \sigma_{-2it,0}(p^m) = \sum_{ab=p^m} \frac{\mathbb{N}(a)^{it}}{\mathbb{N}(b)}.$$

Remember $\tau_{it}(p^m) = \sum_{ab=p^m} \chi_D(a)(a/b)^{it}$, Assume p inert prime, then we claim

$$(10.13) \quad \sum_{\substack{r \in \mathbb{N} \\ r|l}} \tau_{it}\left(\frac{ll'}{r^2}\right) = \sum_{j=0}^m \tau_{it}\left(\frac{p^{2m}}{p^{2j}}\right) = \sum_{ab=p^m} \frac{\mathbb{N}(a)^{it}}{\mathbb{N}(b)}.$$

Let $X = p^{it}$, then the LHS of (10.13) equals

$$\begin{aligned} \frac{1}{X^{2m}} \sum_{j=0}^m X^{2j} \sum_{k=0}^{2(m-j)} (-1)^k X^{2k} &= \frac{1}{X^{2m}} \sum_{j=0}^m X^{2j} \left(\frac{1 + (-1)^{2(m-j)} X^{2(2(m-j)+1)}}{1 + X^2} \right) \\ &= \frac{1}{X^{2m}(1 + X^2)} \left(\frac{1 - X^{2(m+1)}}{1 - X^2} + \frac{X^{2(2m+1)}(1 - X^{-2(m+1)})}{(1 - X^{-2})} \right) \\ &= \frac{1}{X^{2m}(1 + X^2)} \left(\frac{1 - X^{2(m+1)}}{1 - X^2} + \frac{X^{2m+2}(X^{2(m+1)} - 1)}{X^2(1 - X^{-2})} \right) \\ &= \frac{1 - X^{4(m+1)}}{X^{2m}(1 - X^4)}. \end{aligned}$$

The RHS of (10.13) equals

$$\sum_{ab=p^m} \frac{\mathbb{N}(a)^{it}}{\mathbb{N}(b)} = \sum_{k=0}^m X^{2(2k-m)} = \frac{1 - X^{4(m+1)}}{X^{2m}(1 - X^4)},$$

so we are done.

For $p = p_1 p_2$, we show for $l = p_1^m$,

$$(10.14) \quad \sum_{\substack{r \in \mathbb{N} \\ r|l}} \tau_{it}\left(\frac{ll'}{r^2}\right) = \sum_{j=0}^m \tau_{it}(p^m) = \sum_{ab=p_1^m} \frac{\mathbb{N}(a)^{it}}{\mathbb{N}(b)}.$$

Let $X = p^{it}$, then its clear both sides of (10.14) equals

$$X^{-2m} \sum_{j=0}^m X^{2j}.$$

For $p = \sqrt{D}$, the identity is as transparent as the split prime case.

The identity

$$\omega_\mu(l)^{-2} \sigma_{0,\omega_\mu^2}(l) = \sum_{\substack{r \in \mathbb{N} \\ r|l}} \psi_\mu\left(\frac{ll'}{r}\right)$$

is proven in an identical fashion. □

Now using Lemma 10.5 and the duplication formula for the Gamma function, (10.12) equals

$$(10.15) \quad 2\pi(1 + \frac{1}{D}) \sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{\mu} \frac{h(V_1, \mu)h(V_2, \mu)\psi_{\mu}(\frac{l'}{r^2})\overline{\psi_{\mu}(1)}}{L(1, \chi_D)L(1, \omega_{\mu}^2)} +$$

$$2\pi(1 + \frac{1}{D}) \sum_{\substack{r \in \mathbb{N} \\ r|l}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(V_1, t)h(V_2, t)\tau_{it}(\frac{l'}{r^2})\overline{\tau_{it}(1)}}{|L(1 - 2it, \chi_D)|^2} dt + O(X^{-M}).$$

This completes Theorem 1.1 (2.).

Now it can be seen that the continuous spectrum part of (10.11) equals the LHS of (10.12).

Putting all this together we get

$$(10.16) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_{\Pi}) c_n(\Pi) \overline{c_l(\Pi)} + 2\pi(1 + \frac{1}{D}) \left(\sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{\mu} \frac{h(V_1, \mu)h(V_2, \mu)\psi_{\mu}(\frac{l'}{r^2})\overline{\psi_{\mu}(1)}}{L(1, \chi_D)L(1, \omega_{\mu}^2)} + \right.$$

$$\left. \sum_{\substack{r \in \mathbb{N} \\ r|l}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{h(V_1, t)h(V_2, t)\tau_{it}(\frac{l'}{r^2})\overline{\tau_{it}(1)}}{|L(1 - 2it, \chi_D)|^2} dt \right) =$$

$$2\pi(1 + \frac{1}{D}) \sum_{\substack{r \in \mathbb{N} \\ r|l}} \left(\sum_{\phi_{t,D}} M(t_{\phi}) a_{\frac{l'}{r^2}}(\phi_{t,D}) \overline{a_1(\phi_{t,D})} + \right.$$

$$\left. \frac{1}{4\pi} \int_{-\infty}^{\infty} M(t) \eta(\frac{l'}{r^2}, 1/2 + it) \overline{\eta(1, 1/2 + it)} dt + \sum_{\phi_{k,D}} M(k_{\phi}) a_{\frac{l'}{r^2}}(\phi_k) \overline{a_1(\phi_k)} \right) + O(X^{-M})$$

Subtracting the continuous spectrums from both sides we get an equality of the discrete spectrum plus the μ -sum.

10.2. Addressing the Maass constructed theta forms. The μ sum parametrizes the theta forms. Remember from Section 9, $\mu = \frac{k\pi}{\log \epsilon_0}, k \neq 0 \in \mathbb{Z}$. Each $k \neq 0$ corresponds to a cusp form constructed from Hecke characters over the quadratic field \mathbf{K} . The Fourier coefficients of these forms are $\psi_{\mu}(w) := \sum_{\mathbb{N}(q)=w} \omega_{\mu}(q)$, with the corresponding form being

$$\theta_{\omega_{\mu}}(z) = \sum_{n=1}^{\infty} \psi_{\mu}(n) \sqrt{y} K_{\frac{i\pi k}{2 \log \epsilon_0}}(2\pi n y) e(nx).$$

Further $\theta_{\omega_{\mu}} \in \Gamma_0(D, \chi_d)$. For a more explicit explanation of these forms see [Bu]. The cuspidal theta forms base change to Eisenstein series. So it is to be expected that if the Asai L-function is detecting the cuspidal automorphic forms over the quadratic field \mathbf{K} that are base changes, they could not come from theta forms over \mathbb{Q} . These are also the forms which give the poles of the symmetric square L-function in [V].

10.3. **Cuspidal spectrum.** Now with the μ -sum in context, (10.16) equals

$$(10.17) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) \overline{c_l(\Pi)} =$$

$$2\pi(1 + \frac{1}{D}) \sum_{\substack{r \in \mathbb{N} \\ r|l}} \left(\sum_{\phi_{t,D}} h(V_1, t_\phi) h(V_2, t_\phi) a_{\frac{ll'}{r^2}}(\phi_{t,D}) \overline{a_1(\phi_{t,D})} - \sum_{\substack{k \neq 0 \\ k \in \mathbb{Z}}} \frac{h(V_1, \frac{k\pi}{\log \epsilon_0}) h(V_2, \frac{k\pi}{\log \epsilon_0}) \psi_{\frac{k\pi}{\log \epsilon_0}}(\frac{ll'}{r^2}) \overline{\psi_{\frac{k\pi}{\log \epsilon_0}}(1)}}{L(1, \chi_D) L(1, \omega_{\frac{k\pi}{\log \epsilon_0}}^2)} \right)$$

$$+ 2\pi(1 + \frac{1}{D}) \sum_{\phi_{k,D}} h(V_1, k_\phi) h(V_2, k_\phi) a_{\frac{ll'}{r^2}}(\phi_k) \overline{a_1(\phi_k)} + O(X^{-M}),$$

for any integer $M \geq 0$.

But we must account for orthonormalization on the spectral side of the trace formula over \mathbb{Q} . For a theta form the inner product is

$$\langle \psi_\mu, \psi_\mu \rangle = \frac{4\pi L(1, \text{sym}^2(\psi_\mu))}{\cosh(\pi t_j)} = \frac{4\pi L(1, \chi_D) L(1, \omega_\mu^2)}{\cosh(\pi t_j)}.$$

On the other hand, the Fourier coefficients are normalized in the Kuznetsov trace formula. Specifically, a Maass form with eigenvalue $1/4 + t_j^2$, has its coefficient $\rho(n)$ normalized by the factor

$$\left(\frac{4\pi|n|}{\cosh(\pi t_j)} \right)^{1/2}$$

(see [H], [Iw]). Let

$$\Psi_{\frac{k\pi}{\log \epsilon_0}}(n) := \left(\frac{4\pi|n|}{\cosh(\frac{k\pi}{\log \epsilon_0})} \right)^{1/2} \psi_{\frac{k\pi}{\log \epsilon_0}}(n),$$

then

$$(10.18) \quad \sum_{\substack{k \neq 0 \\ k \in \mathbb{Z}}} \frac{h(V_1, \frac{k\pi}{\log \epsilon_0}) h(V_2, \frac{k\pi}{\log \epsilon_0}) \psi_{\frac{k\pi}{\log \epsilon_0}}(\frac{ll'}{r^2}) \overline{\psi_{\frac{k\pi}{\log \epsilon_0}}(1)}}{L(1, \chi_D) L(1, \omega_\mu^2)} =$$

$$= \sum_{\substack{k \neq 0 \\ k \in \mathbb{Z}}} \frac{1}{\langle \psi_\mu, \psi_\mu \rangle} h(V_1, \frac{k\pi}{\log \epsilon_0}) h(V_2, \frac{k\pi}{\log \epsilon_0}) \Psi_{\frac{k\pi}{\log \epsilon_0}}(\frac{ll'}{r^2}) \overline{\Psi_{\frac{k\pi}{\log \epsilon_0}}(1)}.$$

The theta forms coming from the continuous spectrum calculation from section 9 over the quadratic field \mathbf{K} now cancels with theta forms coming from the spectral side of the trace formula over \mathbb{Q} .

Therefore, we find (10.16) implies

$$(10.19) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) \overline{c_l(\Pi)} = 2\pi(1 + \frac{1}{D}) \left(\sum_{\substack{r \in \mathbb{N} \\ r|l}} \sum_{\phi_{t,D} \neq \theta_{\omega_\mu}} M(t_\phi) a_{\frac{ll'}{r^2}}(\phi_{t,D}) \overline{a_1(\phi_{t,D})} + \right.$$

$$\left. \sum_{\phi_{k,D}} M(k_\phi) a_{\frac{ll'}{r^2}}(\phi_k) \overline{a_1(\phi_k)} \right) + O(X^{-M}).$$

This completes Theorem 1.1 (1.).

11. HECKE ALGEBRA

We now get matching of the individual representations. We exploit the fact that each irreducible representation Π has associated Fourier coefficients that obey Hecke relations:

$$c_n(\Pi)c_{m_1}(\Pi) = \sum_{r|(\mathbf{n}, m_1)} c_{\frac{m_1 \mathbf{n}}{r^2}}(\Pi),$$

one gets

$$(11.1) \quad \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_\mu(\Pi) c_q(\Pi) c_\nu(\Pi) = \sum_{r|(q, \nu)} \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_\mu(\Pi) c_{\frac{q\nu}{r^2}}(\Pi).$$

Likewise, one gets

$$(11.2) \quad \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_\mu(\Pi) \left[\prod_{i=1}^N c_{q_i}(\Pi) \right] c_\nu(\Pi) = \sum_{r_1|(q_1, \nu)} \sum_{r_2|(\frac{q_1 \nu}{r_1^2}, q_2)} \sum_{r_3|(\frac{q_1 q_2 \nu}{(r_1 r_2)^2}, q_3)} \dots \sum_{r_N|(\frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2}, q_N)} \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_\mu(\Pi) c_{\frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2}}(\Pi).$$

So one computes by using Theorem 1.1 and (11.2) for any q_i such that $(\prod_{i=1}^N q_i, D) = 1$,

$$(11.3) \quad \frac{1}{X} \sum_{\mathbf{m}} \sum_n g(\mathbf{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) \left[\prod_{i=1}^N c_{q_i}(\Pi) \right] c_l(\Pi) = \sum_{r_1|(q_1, \nu)} \sum_{r_2|(\frac{q_1 \nu}{r_1^2}, q_2)} \sum_{r_3|(\frac{q_1 q_2 \nu}{(r_1 r_2)^2}, q_3)} \dots \sum_{r_N|(\frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2}, q_N)} \frac{1}{X} \sum_{\mathbf{m}} \sum_n g(\mathbf{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) c_{\frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2}}(\Pi) = 2\pi(1 + \frac{1}{D}) \left(\sum_{\substack{r \in \mathbb{N} \\ r | \frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2}}} \sum_{\phi_{t,D} \neq \theta_{\omega_\mu}} h(V, t_\phi) a_{\frac{\mathbf{N}(\frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2})}{r^2}}(\phi_{t,D}) \overline{a_1(\phi_{t,D})} + \sum_{\phi_{k,D}} h(V, k_\phi) a_{\frac{\mathbf{N}(\frac{\prod_{i=1}^N q_i \nu}{\prod_{i=1}^N r_i^2})}{r^2}}(\phi_k) \overline{a_1(\phi_k)} \right) + O(X^{-M})$$

In particular, for any polynomial $F(c_{q_1}, c_{q_2}, \dots, c_{q_N})$ with complex coefficients

$$(11.4) \quad \frac{1}{X} \sum_{\mathbf{m}} \sum_n g(\mathbf{m}^2 n / X) \sum_{\Pi \neq 1} h(V, \nu_\Pi) c_n(\Pi) F(c_{q_1}, c_{q_2}, \dots, c_{q_N}) \overline{c_l(\Pi)}$$

has a corresponding identity like Theorem 1.1 over the forms of level D on \mathbb{Q} . Call the term corresponding to $F(c_{q_1}, c_{q_2}, \dots, c_{q_N})$ on the side of forms over \mathbb{Q} , $T \circ F$.

With this freedom, if we were in a finite dimensional setting (e.g. the space of forms of a fixed weight k or eigenvalue parameter t_j), we can choose such a polynomial F to be zero on all but one single representation Π .

11.1. Reduction to a finite dimensional setting. Now we use Hypothesis 1.2 in the following propositions of [V]:

Proposition 11.1. *Let t_j be a discrete subset of \mathbb{R} with $\{j : t_j \leq T\} \ll T^r$ for some r . Let, for each j , there be given a function $c_X(t_j)$ depending on X , so that $c_X(t_j) \ll t_j^{r'}$ for some*

r' - the implicit constant independent of X ; similarly, for each k odd, let there be given a function $c_X(k)$ depending on X so that $c_X(k) \ll k^{r'}$. Suppose that one has an equality

$$(11.5) \quad \lim_{X \rightarrow \infty} \left(\sum_j c_X(t_j) h(V, t_j) + \sum_{k \text{ odd}} c_X(k) h(V, k) \right) = 0$$

for all $(h(V, t_j), h(V, k))$ that correspond via Sears-titchmarsh inversion to V . Then $\lim_{X \rightarrow \infty} c_X(t_j)$ exists for each t_j and equals 0, and similarly the same holds for $\lim_{X \rightarrow \infty} c_X(k)$. This equality holds for all functions h for which both sides converge.

Proposition 11.2. *Given $j_0 \in \mathbb{N}, \epsilon > 0$ and an integer $N > 0$, there is a V of compact support so that $h(V, t_j) = 1$, and for all $j' \neq j_0$, $h(V, t_{j'}) \ll \epsilon(1 + |t_{j'}|)^{-N}$, and for all k odd, $h(V, k) \ll \epsilon k^{-N}$.*

Given $k_0, \epsilon > 0$ and an integer $N > 0$, there is a V of compact support so that $h(v, k_0) = 1$, $h(V, k) \ll \epsilon k^{-N}$ for k odd $k \neq k_0$, and $h(V, t) \ll (1 + |t|)^{-N}$ for all \mathbb{R} .

Specifically, we let

$$(11.6) \quad c_X(t_j) = \frac{1}{X} \sum_{\substack{\Pi \\ t_\Pi = t_j}} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) c_n(\Pi) F(c_{q_1}, c_{q_2}, \dots, c_{q_N}) \overline{c_l(\Pi)} - \\ 2\pi(1 + \frac{1}{D})(T \circ F) \sum_{\substack{r \in \mathbb{N} \\ r|l}} a_{\frac{ll'}{r^2}}(\phi_{t_j, D}) \overline{a_1(\phi_{t_j, D})} + O(X^{-M}),$$

for any polynomial $F \in \mathbb{C}[x_1, \dots, x_N]$.

Now choosing F to isolate a single Π with archimedean parameter t_j reduces the problem to

$$(11.7) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) c_n(\Pi) \overline{c_l(\Pi)} = 2\pi(1 + \frac{1}{D}) \sum_{\substack{\phi_{t_j, D} \\ t = t_j}} (T \circ F) \sum_{\substack{r \in \mathbb{N} \\ r|l}} a_{\frac{ll'}{r^2}}(\phi_{t_j, D}) \overline{a_1(\phi_{t_j, D})} + O(X^{-M}).$$

If we assume the forms over \mathbb{Q} are Hecke eigenforms, then it is easy to check that

$$(11.8) \quad (T \circ F) = F\left(\sum_{\substack{r \in \mathbb{N} \\ r|q_1}} a_{\frac{\mathbf{N}(q_1)}{r^2}}(\phi_{t_j, D}), \sum_{\substack{r \in \mathbb{N} \\ r|q_2}} a_{\frac{\mathbf{N}(q_2)}{r^2}}(\phi_{t_j, D}), \dots, \sum_{\substack{r \in \mathbb{N} \\ r|q_N}} a_{\frac{\mathbf{N}(q_N)}{r^2}}(\phi_{t_j, D})\right).$$

Furthermore, also using the Hecke qualities of $a_n(\phi_{t_j, D})$, it is easy to check, though laborious, that

$$\alpha_l(\phi_{t_j, D}) := \sum_{\substack{r \in \mathbb{N} \\ r|l}} a_{\frac{\mathbf{N}(l)}{r^2}}$$

has the properties of a Hecke eigenvalue for representations Π over \mathbf{K}/\mathbb{Q} with archimedean parameter $t_\Pi = \{t_j, t_j\}$.

Therefore, with (11.8), the polynomial F kills all but at most one of the terms $\phi_{t_j, D}$ on the RHS of (11.7). If it kills all of them, this implies by the LHS of (11.7) that if $\Pi \neq 0$ there exists an l such that $c_l(\Pi) \neq 0$, and

$$\frac{1}{X} \sum_{n, m} g\left(\frac{nm^2}{X}\right) c_n(\Pi) = O(X^{-M}),$$

for any $M > 0$. However, this contradicts the assumption that the Asai L-function has a pole at $s = 1$.

So there is exactly one term on the RHS of (11.7) and so we have

$$(11.9) \quad \frac{1}{X} \sum_{\mathfrak{m}} \sum_n g(\mathfrak{m}^2 n / X) c_n(\Pi) \overline{c_l(\Pi)} = 2\pi \left(1 + \frac{1}{D}\right) \sum_{\substack{r \in \mathbb{N} \\ r|l}} a_{\frac{w}{r^2}}(\phi_{t_j, D}) + O(X^{-M}),$$

Hence if we let $l = 1$, we have

$$(11.10) \quad \frac{1}{X} \sum_{n, m} g\left(\frac{nm^2}{X}\right) c_n(\Pi) = 2\pi \left(1 + \frac{1}{D}\right) + O(X^{-M}).$$

But since (11.9) holds for all $l \in \mathcal{O}$, $(l, D) = 1$, we must have

$$c_l(\Pi) = \sum_{\substack{r \in \mathbb{N} \\ r|l}} a_{\frac{w}{r^2}}(\phi_{t_j, D}).$$

This proves Corollary 1.3.

By Mellin inversion

$$(11.11) \quad \frac{1}{X} \sum_{n, m} g\left(\frac{nm^2}{X}\right) c_n(\Pi) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{g}(s) X^{s-1} \zeta(2s) L(s, \Pi, Asai) ds = \delta(\Pi) \hat{g}(1) \zeta(2) \text{Res}_{s=1} L(s, \Pi, Asai) + O(X^{-M}),$$

where $\delta(\Pi) = 1$ if Π is a base change and 0 else. The last equality comes from shifting the contour to the left to $-M$. Since M is arbitrary, $\zeta(2s) L(s, \Pi, Asai)$ has analytic continuation to the entire plane with at most a simple pole at $s = 1$. This concludes Corollary 1.4.

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